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CONTENTS

The Mathematical Theory of the Extensional Vibration of a Bar excited by the Impact of an Elastic Load

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Introduction.

The dynamical theory of impact of a rigid load striking longitudinally at the free end of a bar, the other end being fixed, has long been worked out by Boussinesq * and others, with the help of St. Venant's method of 'variation of integration constant.' In a previous paper † we have made suggestions how this theory can be extended to the case of an elastic load which ordinarily obeys Hooke's law of compression, throughout the period of its contact, with the free end of the bar. In order to explain Tschudi's ‡ observation about the dependence of duration of contact on velocity of impact and also the fluctuating nature of the pressure during contact, we have assumed § the load to be plastic and have divided the total duration of impact into three successive sub periods, as Andrews ‖ has done in his treatment of the problem of collision between two similar balls. During the first and the last sub periods each of which being equal to τ, the mechanism of impact is assumed to be governed by Hertz's law. This is discussed in our previous paper § in detail. The impact, during

* Boussinesq—Application des potentiels, Paris (1885). Love—The Mathematical Theory of Elasticity (4th edition), art. 281, pp. 431-441. The references to the other earlier workers are given in the introduction of this treatise, pp. 26-27.

† Ghosh—Ind. Phy. Math. Jour., Vol. 3, pp. 73-79 (1932). Approved Thesis for the Griffith Memorial Prize of the Calcutta University.

‡ Tschudi—Phy. Rev., Vol. 18, p. 423 (1921), Vol. 23, p. 956 (1924).

§ Ghosh—Zeit. f. Angw. Math. Mec., Vol. 14, pp. 71-76 (1934).

‖ Andrews—Phil. Mag., Vol. 8, p. 781 (1929); Vol. 9, p. 593 (1930). Proc. Phy. Soc., Vol. 43, p. 1 (1931)—Approved Doctorate Thesis of the London University.

the intermediate sub period, obeys Hooke's law of compression and waves are generated within the bar from the struck end

In this paper, we propose to work out the problem in detail for Hooke's sub period, which is composed of a number of small intervals In Section I, we shall extend the theory following St. Venant's principle. In doing so, we shall adopt the symbolic representation of the differential operator, in order to overcome the difficulty arising out of the integration at successive stages In Section II, we shall give a generalised treatment of the above problem. It simplifies the process of successive deductions for different intervals In Section III, we shall give the general expression for the displacement and pressure In Section IV we shall consider the special case of the rigid load for higher intervals. And in Section V we shall deduce the expression for the duration of contact in the case of an elastic load

Section I.

The differential equation of the extentional vibration is

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial s^2}, \qquad \ldots \quad (1)$$

where s is measured from the fixed end of the bar, w is the longitudinal displacement, c the velocity of the longitudinal wave propagation along the bar, given by $c^2 = E_1 a/\rho$, E_1 being Young's modulus, a the cross section, ρ the mass per unit length of the bar.

The bar being fixed at $s=0$, the value of w is zero at the point The terminal conditions at $s=l$, l being the length of the bar, is given by the equation of the motion of the striking body, which is also supposed to be elastic, or it is for Hooke's period

$$P = M\left(\frac{\partial^2 z}{\partial t^2}\right) = -E_1 a \left(\frac{\partial w}{\partial s}\right)_{s=l}$$

$$= -E_2 \xi \qquad \text{(Hooke's law)}, \qquad \ldots \quad (2)$$

where E_2 is the elastic constant depending on the material of the load and its shape and size, z the displacement of the centre of gravity of the load is given by

$$z = w_{s=l} + \xi, \qquad \ldots \quad (3)$$

P and ξ represent pressure and compression of the load, and is measured from the beginning of the Hooke's period, that is, $t=\tau$

The solution of (1) is of the form

$$w=F(ct-s)+\psi(ct+s), \qquad \dots \quad (4)$$

where F and ψ are arbitrary functions The terminal condition $w=0$ at $s=0$ reduces the eq (4) to the form

$$w=F(ct-s)-F(ct+s), \qquad \dots \quad (5)$$

From eq. (2) with the help of (5) we have,

$$\xi=-\lambda[F'(ct-l)+F'(ct+l)], \qquad \dots \quad (6a)$$

where

$$\lambda=E_1 a/E_2. \qquad \dots \quad (6b)$$

Now we may consider F a function of an argument ζ which may be put equal to $(ct-s)$ or $(ct+s)$ when required.

On substituting the value of ξ from (6a) and w from (4) in eq. (2), we have

$$Mc^2\{F''(ct-l)-F''(ct+l)\}-Mc^2\lambda\{F'''(ct-l)+F'''(ct+l)\}$$
$$=E_1 a\{F'(ct-l)+F'(ct+l)\}. \qquad \dots \quad (7)$$

which is the "Equation promotrice" obtained from the terminal condition at $s=l$ or

$$F'''(\zeta)+\frac{1}{\lambda}F''(\zeta)+\frac{E_2}{Mc^2}F'(\zeta)=\frac{2}{\lambda}F''(\zeta-2l)-\Big\{F'''(\zeta-2l)$$
$$+\frac{1}{\lambda}F''(\zeta-2l)+\frac{E_2}{Mc^2}F'(\zeta-2l)\Big\}. \qquad \dots \quad (8)$$

The complete integral of eq. (8) is

$$F'(\zeta)=Ae^{q\zeta}+Be^{p\zeta}+\frac{2}{\lambda}\frac{F''(\zeta-2l)}{f(D)}-F'(\zeta-2l) \qquad \dots \quad (9)$$

where q and p are the roots of the equation

$$f(D)\equiv D^2+\frac{1}{\lambda}D+\frac{E_2}{Mc^2}=0, \qquad \dots \quad (10)$$

and are given by

$$[q, p] = -\frac{E_2}{2E_1 a} \pm \frac{1}{2}\sqrt{\left\{\frac{E_2^2}{E_1^2 a^2} - \frac{4E_2}{Mc^2}\right\}} \qquad \dots \quad (11)$$

and A, B are constants of integration When $3l > \zeta - c\tau > l$, the expression $\frac{2}{\lambda} \cdot \frac{F''(\zeta-2l)}{f(D)} - F'(\zeta-2l)$ vanishes as $F(\zeta-2l)$ is known only from the interval $5l > \zeta - c\tau > 3l$. So $F'(\zeta)$, during $3l > \zeta - c\tau > l$ reduces to

$$F'(\zeta) = Ae^{q\zeta} + Be^{p\zeta} \qquad \dots \quad (12)$$

From the boundary conditions, namely at $t=\tau$, $\zeta=0$, $\dot{\iota} = -v_1$, the velocity of the load at the beginning of the Hooke's period, we have from eq (5) and (6a)

$$F'(c\tau - l + 0) + F'(c\tau + l + 0) = 0 \qquad \dots \quad (13a)$$

and

$$c[F'(c\tau - l + 0) - F'(c\tau + l + 0) - \lambda\{F''(c\tau - l + 0) + F''(c\tau + l + 0)\}] = -v_1 \qquad \dots \quad (13b)$$

Or we have

$$\left.\begin{aligned} F'(c\tau + l + 0) &= 0 \\ F''(c\tau + l + 0) &= \frac{v_1}{c\lambda} \end{aligned}\right\} \qquad \dots \quad (14)$$

which, with the help of eq. (12) lead to

$$\left.\begin{aligned} Ae^{q(c\tau+l)} + Be^{p(c\tau+l)} &= 0 \\ Aqe^{q(c\tau+l)} + Bpe^{p(c\tau+l)} &= \frac{v_1}{c\lambda} \end{aligned}\right| \qquad \dots \quad (15)$$

On solving eq. (15) for A and B, we get

$$A = \frac{v_1}{c\beta}e^{-q(c\tau+l)}, \quad B = -\frac{v_1}{c\beta}e^{-p(c\tau+l)}, \qquad \dots \quad (16a)$$

where $\beta = \lambda(q-p)$ and $\lambda = E_1 a/E_2 = -\frac{1}{(q+p)}$, $\qquad \dots \quad (16b)$

Thus $F'(\zeta)$, during the interval $3l>\zeta-c\tau>l$, with the help of (16a), eq. (12) can be written in the form

$$F'(\zeta)=\frac{v_1}{c\beta}(e^{q\zeta_1}-e^{p\zeta_1}), \qquad \dots \quad (17a)$$

where $$\zeta_1=\zeta-c\tau-l \qquad \dots \quad (17b)$$

When $5l>\zeta-c\tau>3l$, we have from (17)

$$F''(\zeta-2l)=\frac{v_1}{c\beta}[qe^{q(\zeta-c\tau-3l)}-pe^{p(\zeta-c\tau-3l)}] \quad \dots \quad (18a)$$

So, from the eq (9) we have for $F'(\zeta)$ during this interval

$$F'(\zeta)=Ae^{q\zeta}+Be^{p\zeta}+\frac{2v_1}{c\beta\lambda}\cdot\frac{1}{f(D)}[qe^{q\zeta_2}-pe^{p\zeta_2}]$$

$$-\frac{v_1}{c\beta}[e^{q\zeta_2}-e^{p\zeta_2}], \qquad \dots \quad (18b)$$

where $$\zeta_2=\zeta-c\tau-3l=\zeta_1-2l. \qquad \dots \quad (18c)$$

Now if q and p are the roots of the eqation $f(D)=0$, we have,

$$\left.\begin{aligned}\frac{e^{qx}.x^m}{f(D)}&=\frac{e^{qx}}{(q-p)}\Big[\frac{1}{D}-\frac{1}{(q-p)}+\frac{D}{(q-p)^2}-\frac{D^2}{(q-p)^3}+\dots\\&\qquad+(-1)^m\frac{D^{m-1}}{(q-p)^m}\Big]x^m,\\ \text{and}\quad&\\ \frac{e^{px}.x^m}{f(D)}&=-\frac{e^{px}}{(q-p)}\Big[\frac{1}{D}+\frac{1}{(q-p)}+\frac{D}{(q-p)^2}+\frac{D^2}{(q-p)^3}+\dots\\&\qquad+\frac{D^{m-1}}{(q-p)^m}\Big]x^m,\end{aligned}\right\} \quad \dots \quad (19)$$

where D and $\frac{1}{D}$ have got their usual meanings.

Thus eq. (18) with the help of (19) reduces to

$$F'(\zeta)=Ae^{q\zeta}+Be^{p\zeta}+\frac{2v_1}{c\beta^2}\left[e^{q\zeta_2}q\zeta_2+e^{p\zeta_2}p\zeta_2\right]$$

$$-\frac{v_1}{c\beta}\left[e^{q\zeta_2}-e^{p\zeta_2}\right]. \quad \ldots \quad (20)$$

The condition of continuity of ξ and $\dot{z}$ at $c(t-\tau)=2l$ give,

$$F'(c\tau+l-0)+F'(c\tau+3l-0)=F'(c\tau+l+0)+F'(c\tau+3l+0) \ldots \quad (21a)$$

and

$$F'(c\tau+l-0)-F'(c\tau+3l-0)-\lambda\{F''(c\tau+l-0)+F''(c\tau+3l-0)\}$$

$$=F'(c\tau+l+0)-F'(c\tau+3l+0)-\lambda\{F''(c\tau+l+0)+F''(c\tau+3l+0)\} \ldots (21b)$$

From eqs (17), (20) and (21), we get

$$Ae^{q(c\tau+3l)}+Be^{p(c\tau+3l)}=\frac{v_1}{c\beta}\left[e^{q2l}-e^{p2l}\right], \quad \ldots \quad (22a)$$

$$Aqe^{q(c\tau+3l)}+Bpe^{p(c\tau+3l)}=\frac{v_1}{c\beta}\left[qe^{q2l}-pe^{p2l}\right]-\frac{2v_1}{c\beta^2}(q+p). \quad (22b)$$

On solving eq (22) for A and B, we get

$$A=\frac{v_1}{c\beta}e^{-q(c\tau+l)}+\frac{2v_1}{c\beta^3}e^{-q(c\tau+3l)},$$

$$B=-\frac{v_1}{c\beta}e^{-p(c\tau+l)}-\frac{2v_1}{c\beta^3}e^{-p(c\tau+3l)}.$$

Hence, from (23), $F'(\zeta)$ during the interval $5l>\zeta-c\tau>3l$ becomes,

$$F'(\zeta)=\frac{v_1}{c\beta}\left[e^{q\zeta_1}-e^{p\zeta_1}\right]+\frac{v_1}{c\beta^3}\left[e^{q\zeta_2}\{2-\beta^2+2\beta q\zeta_2\}\right.$$

$$\left.-e^{p\zeta_2}\{2-\beta^2-2\beta p\zeta_2\}\right], \quad \ldots \quad (24)$$

where $\zeta_1=\zeta-c\tau-l$ and $\zeta_2=\zeta_1-2l=\zeta-c\tau-3l$

When $7l>\zeta-c\tau>5l$, we have, from (24)

$$F''(\zeta-2l)=\frac{v_1}{c\beta}\left[qe^{q\zeta_2}-pe^{p\zeta_2}\right]+\frac{v_1}{c\beta^3}\left[(1+2\beta-\beta^2)e^{q\zeta_3}\right.$$

$$\left.-(1-2\beta-\beta^2)e^{p\zeta_3}\right]+\frac{2v_1}{c\beta^2}\left[qe^{q\zeta_3}\,q\zeta_3+pe^{p\zeta_3}\,p\zeta_3\right]\ldots\quad(25a)$$

where $\zeta_3'=\zeta_2-2l=\zeta_1-4l$ ... (25)

The value for the expression $\frac{2}{\lambda}\,\frac{F''(\zeta-2l)}{f(D)}$, as occurring in eq. (9) can very easily be obtained from (25a) and the general relation (19), as,

$$\frac{2}{\lambda}\,\frac{F''(\zeta-2l)}{f(D)}=\frac{2v_1}{c\beta^2}\left[e^{q\zeta_3}\,q\zeta_3+e^{p\zeta_3}\,p\zeta_3\right]$$

$$+\frac{v_1}{c\beta^3}\left[e^{q\zeta_3}\{(1-\beta)^2+(3+\beta-\beta^2)2\beta q\zeta_3+\tfrac{1}{2}(2\beta q\zeta_3)^2\}\right.$$

$$\left.-e^{p\zeta_3}\{(1+\beta)^2-(3-\beta-\beta^2)2\beta p\zeta_3+\tfrac{1}{2}(2\beta p\zeta_3)^2\}\right].\quad(26)$$

So $F'(\zeta)$ during this interval can be readily obtained from eq. (9) and (26) as follows

$$F'(\zeta)=Ae^{q\zeta}+Be^{p\zeta}+\frac{2v_1}{c\beta^2}\left[e^{q\zeta_3}\cdot q\zeta_3+e^{p\zeta_3}p\zeta_3\right]-\frac{v_1}{c\beta}\left[e^{q\zeta_3}-e^{p\zeta_3}\right]$$

$$+\frac{v_1}{c\beta^3}\left[e^{q\zeta_3}\{(1-\beta)^2+(3+\beta-\beta^2)2\beta q\zeta_3+\tfrac{1}{2}(2\beta q\zeta_3)^2\}\right.$$

$$\left.-e^{p\zeta_3}\{(1+\beta)^2-(3-\beta-\beta^2)2\beta p\zeta_3+\tfrac{1}{2}(2\beta p\zeta_3)^2\}\right]$$

$$-\frac{v_1}{c\beta^3}\left[e^{q\zeta_3}\{(2-\beta^2)+2\beta q\zeta_3\}-e^{p\zeta_3}\{(2-\beta^2)-2\beta p\zeta_3\}\right]\quad(27)$$

The continuity of ξ and $\dot{z}$ at $c(t-\tau)=4l$, give

$$F'(c\tau+3l-0)+F'(c\tau+5l-0)=F'(c\tau+3l+0)+F'(c\tau+5l+0), \quad \ldots \quad (28a)$$

$$F'(c\tau+3l-0)-F'(c\tau+5l-0)-\lambda\{F''(c\tau+3l-0)+F''(c\tau+5l-0)\}$$

$$=F'(c\tau+3l+0)-F'(c\tau+5l+0)-\lambda\{F''(c\tau+3l+0)+F''(c\tau+5l+0)\} \quad (28b)$$

With the help of (24) and (27), eqs. (28a) and (28b) become,

$$Ae^{q(c\tau+5l)}+Be^{p(c\tau+5l)}=\frac{v_1}{c\beta}\left[e^{q4l}-e^{p4l}\right]$$

$$+\frac{2v_1}{c\beta^3}\left[e^{q2l}-e^{p2l}\right]-\frac{v_1}{c\beta^5}\left[(1-\beta)^2-(1+\beta)^2\right], \ldots \quad (29a)$$

and

$$Aqe^{q(c\tau+5l)}+Bpe^{p(c\tau+5l)}=\frac{v_1}{c\beta}\left[qe^{q4l}-pe^{p4l}\right]$$

$$+\frac{2v_1}{c\beta^3}\left[qe^{q2l}-pe^{p2l}\right]-\frac{v_1}{c\beta^5}\left[(3+\beta-\beta^2)q+(3-\beta-\beta^2)p\right] \;. \quad (29b)$$

On solving

$$A=\frac{v_1}{c\beta}e^{-q(c\tau+l)}+\frac{2v_1}{c\beta^3}e^{-q(c\tau+3l)}+\frac{v_1}{c\beta^5}\left[(3+\beta-\beta^2)(1-\beta)\right.$$

$$\left.+(3-\beta-\beta^2)(1+\beta)-(1-\beta)^2\right]e^{-q(c\tau+5l)},$$

and

$$B=-\frac{v_1}{c\beta}e^{-p(c\tau+l)}-\frac{2v_1}{c\beta^3}e^{-p(c\tau+3l)}-\frac{v_1}{c\beta^5}\left[(3+\beta-\beta^2)(1-\beta)\right.$$

$$\left.+(3-\beta-\beta^2)(1+\beta)-(1+\beta)^2\right]e^{-p(c\tau-5l)}. \quad \ldots \quad (30)$$

Hence, when $\qquad 7l>\zeta-c\tau>5l,$

$$F'(\zeta)=\frac{v_1}{c\beta}[e^{q\zeta_1}-e^{q\zeta_1}]+\frac{v_1}{c\beta^5}[e^{q\zeta_2}(2-\beta^2+2\beta q\zeta_2)$$

$$-e^{p\zeta_2}(2-\beta^2-2\beta p\zeta_2)]+\frac{v_1}{c\beta^5}[e^{q\zeta_3}\{\beta^4+6(1-\beta^2)$$

$$+2\beta(1+\beta)(3-2\beta)q\zeta_3+2\beta^2q^2\zeta_3^2\}-e^{p\zeta_3}\{\beta^4+6(1-\beta^2)$$

$$-2\beta(1-\beta)(3-2\beta)p\zeta_3+2\beta^2p^2\zeta_3^2\}] \quad \ldots \quad (31)$$

where ζ_1 ζ_2 and ζ_3 are given by (17b) (18c) and (25b)

In a similar manner $F'(\zeta)$ for $9l>\zeta-c\tau>7l$ and at intervals higher than that, can very easily be calculated by taking help of eqs (9), (19) and from $F'(\zeta)$ of the previous interval, of course the tedious process of integration is got rid of, in this present method,—by the symbolic representations of the differential operator.

Section II

In this section we shall establish relations between the constant coefficients of the functions, during the interval $(2n-1)l>\zeta-c\tau>(2n-3)l$ and $(2n+1)l>\zeta-c\tau>(2n-1)l$, which will help us to know completely the function during any interval, from a knowledge of the function, during the interval just previous to it.

For the sake of simplicity we put

$$\left.\begin{aligned}
&\zeta-c\tau-l=\zeta_1\\
&\zeta-c\tau-3l=\zeta_1-2l=\zeta_2\\
&\zeta-c\tau-5l=\zeta_2-2l=\zeta_3\\
&\ldots \quad \ldots \quad \ldots \quad \ldots\\
&\zeta-c\tau-(2n-1)l=\zeta_{n-1}-2l=\zeta_n
\end{aligned}\right\} \quad \ldots \quad (1)$$

And in order to avoid confusion, we write $F_1(\zeta)$, $F_2(\zeta)\ldots F_n(\zeta)$ for the function $F(\zeta)$ during the interval

$$3l>\zeta-c\tau>l,\ 5l>\zeta-c\tau>3l,\ldots$$

$$(2n+1)l>\zeta-c\tau>(2n-1)l$$

The study of the results of the previous sections shows that,

$$\left.\begin{aligned}
&F'_1(\zeta)=\phi_1(\zeta_1),\\
&F'_2(\zeta)=\phi_1(\zeta_1)+\phi_2(\zeta_2)=F'_1(\zeta)+\phi_2(\zeta_2),\\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots\\
&F'_n(\zeta)=\phi_1(\zeta_1)+\phi_2(\zeta_2)+\phi_n(\zeta_n)=F'_{n-1}(\zeta)+\phi_n(\zeta_n),
\end{aligned}\right\} \quad \dots \quad (2)$$

where, the form of the functions ϕ can be represented by

$$\phi_1(\zeta_1)=a_{1,0}e^{q\zeta_1}+b_{1,0}e^{p\zeta_1}, \quad \dots \quad (3)$$

$$\phi_2(\zeta_2)=e^{q\zeta_2}(a_{2,0}+a_{2,1}\zeta_2)+e^{p\zeta_2}(b_{2,0}+b_{2,1}\zeta_2),$$

$$=e^{q\zeta_2}\sum_{r=0}^{1}a_{2,r}\zeta_2^r+e^{p\zeta_2}\sum_{r=0}^{1}b_{2,r}\zeta_2^r \quad \dots \quad (4)$$

and

$$\phi_n(\zeta_n)=e^{q\zeta_n}\sum_{r=0}^{(n-1)}a_{n,r}\zeta_n^r+e^{p\zeta_n}\sum_{r=0}^{(n-1)}b_{n,r}\zeta_n^r. \quad \dots \quad (5)$$

The Equation promotrice, represented by eq (8), Sec. I, can be written for different intervals from eq (2) as follows

$$f(D)\phi_1(\zeta_1)=0$$

during $3l>\zeta-c\tau>l,$

whence $f(D)\phi_2(\zeta_2)=\dfrac{2}{\lambda}\phi'_1(\zeta_2)-f(D)\phi_1(\zeta_2),$

during $5l>\zeta-c\tau>3l,$

and

$$f(D)\phi_n(\zeta_n)=\frac{2}{\lambda}\phi'_{n-1}(\zeta_n)-f(D)\phi_{n-1}(\zeta_n)$$

during $(2n+1)l>\zeta-c\tau+(2n-1)l.$

We shall only consider the last one in order to establish relation between ϕ_n and ϕ_{n-1} and then constant coefficients. This equation can be written as

$$f(D)[\phi_n(\zeta_n)+\phi_{n-1}(\zeta_n)]=\frac{2}{\lambda}\,\phi'_{n-1}(\zeta_n) \qquad \ldots \quad (6)$$

Substituting the values of $\phi_n(\zeta_n)$ and $\phi_{n-1}(\zeta_n)$ from (5), and their first and second derivatives as required by eq. (6), and equating the coefficients of $e^{q\zeta_n}\zeta_n^r$ and $e^{p\zeta_n}\zeta_n^r$ on both sides respectively, we get after simplification, remembering q and p are the roots of the equation

$$f(D)\equiv D^2+\frac{D}{\lambda}+\frac{E_1}{Mc^2}=0,$$

$$a_{n,(r+1)}+\frac{r+2}{q-p}a_{n,(r+2)}=\left(\frac{2}{\beta}-1\right)a_{(n-1),(r+1)}+\frac{2q}{\beta}\,\frac{a_{(n-1),r}}{(r+1)}$$

$$-\frac{r+2}{q-p}a_{(n-1),(r+2)}, \qquad \ldots \quad (7)$$

$$b_{n,(r+1)}-\frac{r+2}{q-2}b_{n,(r+2)}=\frac{r+2}{q-p}b_{(n-1),(r+2)}-\left(\frac{2}{\beta}+1\right)b_{(n-1),(r+1)}$$

$$-\frac{2p}{\beta}\,\frac{b_{(n-1),r}}{r+1}. \qquad \ldots \quad (8)$$

The conditions of continuity of ξ and x at $c(t-\tau)=(n-1)2l$, that is, $\zeta=(2n-1)l+c\tau$ give

$$\lambda[F'(c\tau+\overline{2n-3}\,l-0)+F'(c\tau+\overline{2n-1}\,l-0)]$$

$$=\lambda[F'(c\tau+\overline{2n-3}\,l+0)+F'(c\tau+\overline{2n-1}\,l+0)]$$

and

$$F'(c\tau+\overline{2n-3}\,l-0)-F'(c\tau+\overline{2n-1}\,l-0)-\lambda[F''(c\tau+\overline{2n-3}\,l-0)$$

$$+F''(c\tau+\overline{2n-1}\,l-0)]$$

$$=F'(c\tau+\overline{2n-3}l-0)-F'(c\tau+\overline{2n-1}l-0)-\lambda[F''(c\tau+\overline{2n-3}\,l-0)$$

$$+F''(c\tau+\overline{2n-1}l-0)] \quad \dots \quad (10)$$

By the help of the last equation given in (2), eqs. (9) and (10) are reduced to

$$\lambda[\phi_n(0)+\phi_{n-1}(0)]=0 \quad \dots \quad (11)$$

and

$$\lambda[\phi'_n(0)+\phi'_{n-1}(0)]=\phi_n(0)-\phi_{n-1}(0) \quad \dots \quad (12)$$

which, by the help of eqs. (4) and (5), become

$$\lambda[a_{n,0}+b_{n,0}+a_{n-1,0}+b_{n-1,0}]=0 \quad \dots \quad (13)$$

and

$$\lambda[q\{a_{n,0}+a_{(n-1),0}\}+p\{b_{n,0}+b_{(n-1),0}\}+a_{n,1}+b_{n,1}+a_{(n-1),1}$$

$$+b_{(n-1),1}]=(a_{n,0}+b_{n,0})-(a_{(n-1),0}+b_{(n-1),0}) \quad \dots \quad (14)$$

or eqs. (13) and (14) become, when $\lambda\neq 0$

$$(a_{n,0}+a_{(n-1),0})+(b_{n,0}+b_{(n-1),0})=0, \quad \dots \quad (15)$$

and

$$q(a_{n,0}+a_{(n-1),0})+p(b_{n,0}+b_{(n-1),0}=-\Big[\frac{2}{\lambda}(a_{(n-1),0}+b_{(n-1),0})$$

$$+(a_{n,1}+b_{n,1})+(a_{(n-1),0}+b_{(n-1),1})] \quad \dots \quad (16)$$

On solving eqs (15) and (16), we get

$$b_{n,0}+b_{(n-1),0}=-(a_{n,0}+a_{(n-1),0})$$

$$=\frac{1}{\beta}[2(a_{(n-1),0}+b_{(n-1),0})+\lambda(a_{n,1}+b_{n,1})+\lambda(a_{(n-1),1}+b_{(n-1),1})] \quad .. \quad (17)$$

The eqs. (7), (8) and (17) will enable us to evaluate $a_{n,r}$ and $b_{n,r}$ for all values of n and r. It should however be remembered that $a_{n,r}$ and $b_{n,r}$ are zero for all values of r greater than $(n-1)$.

Now putting $r=(n-2)$ in eqs. (7) and (8) we get

$$a_{n,(n-1)}=\frac{2q}{\beta}\ \frac{a_{(n-1),(n-2)}}{(n-1)}=\left(\frac{2q}{\beta}\right)^{n-1}\frac{a_{1,0}}{(n-1)!} \quad \dots \quad (18a)$$

$$b_{n,(n-1)}=\left(-\frac{2p}{\beta}\right)\frac{b_{(n-1),(n-2)}}{(n-1)}=\left(-\frac{2p}{\beta}\right)^{n-1}\frac{b_{1,0}}{(n-1)!}\quad\ldots\quad(18b)$$

as $a_{n,n}=b_{n,n}=0$

Again, putting $s=n-3$ in (7) we get by the help of eq. (18a)

$$a_{n,(n-2)}-\frac{2q}{\beta}\,\frac{a_{(n-1)(n-3)}}{(n-2)}=-\frac{1}{(n-2)!}\left(\frac{2q}{\beta}\right)^{n-2}(\beta^2-\beta-1)\frac{a_{1,0}}{\beta^2}$$

and similarly,

$$a_{(n-1)(n-3)}-\frac{2q}{\beta}\,\frac{a_{(n-2)(n-4)}}{(n-3)}=-\frac{1}{(n-3)!}\left(\frac{2q}{\beta}\right)^{n-3}(\beta^2-\beta-1)\frac{a_{1,0}}{\beta^2}$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$a_{4,2}-\frac{2q}{\beta}\,\frac{a_{3,1}}{2}=-\frac{1}{2!}\left(\frac{2q}{\beta}\right)^{2}(\beta^2-\beta-1)\frac{a_{1,0}}{\beta^2}$$

$$a_{3,1}-\frac{2q}{\beta}\,\frac{a_{2,0}}{1}=-\frac{1}{1!}\left(\frac{2q}{\beta}\right)(\beta^2-\beta-1)\frac{a_{1,0}}{\beta^2}.$$

Now, multiplying first, second, third, etc., of the above equations by $\frac{2q}{\beta}$, $\frac{1}{(n-2)}\left(\frac{2q}{\beta}\right)^2$, $\frac{1}{(n-2)(n-3)}\left(\frac{2q}{\beta}\right)^3$, etc., respectively we get, after addition,

$$a_{n,(n-2)}=\left(\frac{2q}{\beta}\right)^{n-2}\frac{1}{(n-2)!}\left[\,a_{2,0}-\frac{(n-2)}{\beta^2}(\beta^2-\beta-1)a_{1,0}\right],$$

proceeding in the similar manner we have from eqs. (8) and (18b)

$$b_{n,(n-2)}=\left(-\frac{2p}{\beta}\right)^{n-2}\frac{1}{(n-2)!}\left[\,b_{2,0}-\frac{(n-2)}{\beta^2}(\beta^2+\beta-1)b_{1,0}\right].$$

These general expressions will enable us to determine the functions completely at different intervals, from the knowledge of the functions during the interval $3l>\zeta-cr>l$ which is given by (*vide* eq. (17), Sec. 1)

$$\left.\begin{aligned}&F'_1(\zeta)=a_{1,0}e^{q\zeta_1}+b_{1,0}e^{p\zeta_1},\\ \text{where}\quad &a_{1,0}=-b_{1,0}=\frac{v_1}{c\beta}.\end{aligned}\right\}\quad\ldots\quad(21)$$

The function during the interval $5l>\zeta-c\tau>3l$, from eqs (2) and (4) is of the form

$$\left.\begin{aligned} &F'_2(\zeta)=F'_1(\zeta)+e^{q\zeta_2}(a_{2,0}+a_{2,1}\zeta_2)+e^{p\zeta_2}(b_{2,0}+b_{2,1}\zeta_2),\\ &\text{where}\\ &a_{2,1}=\frac{v_1}{c\beta^3}\,2\beta q,\ b_{2,1}=\frac{v_1}{c\beta^3}\,2\beta p\\ &\text{and}\\ &a_{2,0}=-b_{2,0}=\frac{v_1}{c\beta^3}\,(2-\beta^2), \end{aligned}\right\} \quad (22)$$

which are obtained by the help of (17) and (18) for $n=2$ and eq. (21) This is same as obtained otherwise in eq (24), Sec I.

The function during the interval $7\,l>\zeta-c\tau>5l$ is of the form

$$F'_3(\zeta)=F'_2(\zeta)+e^{q\zeta_3}\,(a_{3,0}+a_{3,1}\zeta_3+a_{3,2}\zeta_3^2)$$

$$+e^{p\zeta_3}(b_{3,0}+b_{3,1}\zeta_3+b_{3,2}\zeta_3^2), \quad \ldots \quad (23a)$$

where, the co-efficients are obtained very easily by putting $n=3$ and $r=0$ in eqs (7), (17), (18) and (19) which are as follow

$$\left.\begin{aligned} &a_{3,0}=-b_{3,0}=\frac{v_1}{c\beta^5}\,\{\beta^4+6(1-\beta^2)\},\\ &a_{3,1}=\frac{v_1}{c\beta^5}\,(1+\beta)(3-2\beta)2\beta q,\ b_{3,1}=\frac{v_1}{c\beta^5}\,(1-\beta)(3+2\beta)2\beta p,\\ &a_{3,2}=\frac{v_1}{c\beta^5}\,\frac{(2\beta q)^2}{2!},\quad b_{3,2}=\frac{v_1}{c\beta^5}\,\frac{(2\beta p)^2}{2!}. \end{aligned}\right\} \ldots \quad (23b)$$

These are same as given by eqs. (31), Sec. I.

The function during the interval $9l>\zeta-c\tau>7l$ is represented by

$$F'_4(\zeta)=F'_3(\zeta)+e^{q\zeta_4}(a_{4,0}+a_{4,1}\zeta_4+a_{4,2}\zeta_4^2+a_{4,3}\zeta_4^3)$$

$$+e^{p\zeta_4}(b_{4,0}+b_{4,1}\zeta_4+b_{4,2}\zeta_4^2+b_{4,3}\zeta_4^3) \quad \ldots \quad (24a)$$

where the values of the co-efficients are obtained from the same general relations by putting $n=4$, $r=0$ and from the knowledge of the co-efficients at previous interval. These are obtained as given below

$$\left.\begin{aligned}
a_{4,0} &= -b_{4,0} = \frac{v_1}{c\beta^7}(2-\beta^2)[\beta^4+10(1-\beta^2)],\\
a_{4,1} &= \frac{v_1}{c\beta^7}[\beta^4+\beta(1+\beta)(3-2\beta)(2-\beta)+2(1-\beta^2)(5-\beta)]2\beta q,\\
b_{4,1} &= \frac{v_1}{c\beta^7}[\beta^4-\beta(1-\beta)(3+2\beta)(2+\beta)+2(1-\beta^2)(5+\beta)]2\beta p,\\
a_{4,2} &= \frac{v_1}{c\beta^7}[4-\beta(3\beta+2)]\frac{(2\beta q)^2}{2!},\\
b_{4,2} &= \frac{v_1}{c\beta^7}[4-\beta(3\beta+2)]\frac{(2\beta p)^2}{2!},\\
a_{4,3} &= \frac{v_1}{c\beta^7}\frac{(2\beta q)^3}{3!},\quad b_{4,3}=\frac{v_1}{c\beta^7}\frac{(2\beta p)^3}{3!}.
\end{aligned}\right\}\quad (29b)$$

Thus, in the similar manner, we can very easily evaluate the co-efficients of the functions F′ at an interval $11l>\zeta-c\tau>9l$ and at intervals higher than this, by giving different values to n in the general expressions (7), (8) (17) (18) and (19).

Section III.

In Sec. II we have developed a general method by which the constant co-efficients of the $F'(\zeta)$ at different intervals can very easily be determined when the form of the functions is known. In this section we shall give general method of finding out (*i*) $F(\zeta)$, (*ii*) displacement of the struck end of the bar, (*iii*) displacement at any other point of the bar, and (*iv*) pressure of the load at different intervals.

(*i*) *Evaluation of $F(\zeta)$*

From eqs. (2) and (5) Sec. II, we have

$$F'_n(\zeta)=F'_{n-1}(\zeta)+\phi_n(\zeta_n), \qquad \dots \quad (1a)$$

where

$$\phi_n(\zeta_n)=e^{q\zeta_n}\sum_{r=0}^{n} a_{n,r}\zeta_n^r+e^{p\zeta_n}\sum_{r=0}^{n} b_{n,r}\zeta_n^r. \qquad \dots \quad (1b)$$

Integrating we get

$$F_n(\zeta)=F_{n-1}(\zeta)+\int\phi_n(\zeta_n)d\zeta_n+\text{const} \quad \ldots \quad (2a)$$

as

$$d\zeta=d\zeta_1==d\zeta_n.$$

From (1b)

$$\int\phi_n(\zeta_n)d\zeta_n=\frac{e^{q\zeta_n}}{q}\sum_{r=0}^{n}A_{n,r}\zeta_n^r+\frac{e^{p\zeta_n}}{p}\sum_{r=0}^{n}B_{n,r}\xi_n^r,$$

where

$$A_{n,r}=a_{n,r}-{}^{r+1}P_1\frac{a_{n,(r+1)}}{q}+{}^{r+2}P_2\frac{a_{n,(r+2)}}{q^2}+\ldots\ldots$$

$$+(-1)^{n-r-1}\,{}^{n-1}P_{n-r-1}\frac{a_{n,(n-1)}}{q^{n-r-1}},$$

$$B_{n,r}=b_{n,r}-{}^{r+1}P_1\frac{b_{n,(r+1)}}{p}+{}^{r+1}P_2\frac{a_{n,(r+1)}}{p^2}+\ldots\ldots$$

$$-(-1)^{n-r-1}\,{}^{n-1}P_{n-r-1}\frac{b_{n,(n-1)}}{p^{n-r-1}}. \quad \ldots \quad (3)$$

here P stands for the permutation. The other co-efficients are obtained by giving values 0, 1, 2, ... $(n-1)$ to r. The co-efficients $A_{n,r}$ and $B_{n,r}$ for r greater than $(n-1)$ do not occur.

The constant of integration in $(2a)$ is evaluated from the condition that $F_{n-1}(\zeta)$ and $F_n(\zeta)$ are continuous at $s=l$ and $c(t-\tau)=(n-1)2l$, which gives,

$$\text{const.}=-\frac{A_{n,0}}{q}-\frac{B_{n,0}}{p}. \quad \ldots \quad (4)$$

So the eq. $(2a)$, by the help of $(2b)$ and (4) becomes

$$F_n(\zeta)=F_{n-1}(\zeta)+\left[\frac{e^{q\zeta_n}}{q}\sum_{r=0}^{(n-1)}A_{n,r}\zeta_n^r-\frac{A_{n,0}}{q}+\frac{e^{p\zeta_n}}{p}\cdot\right.$$

$$\left.\sum_{r=0}^{(n-1)}B_{n,r}\zeta_n^r-\frac{B_{n,0}}{p}\right], \quad \ldots \quad (5)$$

where the co-efficients $A_{n,1}A_{n,2}$ etc., are obtained from eq. (3).

Special Cases :

(*a*) $F(\zeta)$ during the interval $3l > \zeta - c\tau > l$

Put $n=1$, we have from eq. (5), as F_0 does not occur,

$$F(\zeta)=\frac{A_{1,0}}{q}(e^{q\zeta_1}-1)+\frac{B_{1,0}}{p}(e^{p\zeta_1}-1) \quad \ldots \quad (6a)$$

where $\quad A_{1,0}=a_{1,0}, \quad B_{1,0}=b_{1,0} \quad \ldots \quad (6b)$

which are obtained by putting $n=1$, $r=0$ in eq. (3)

(*b*) $F(\zeta)$ during the interval $5l > \zeta - c\tau > 3l$

Put $n=2$, we have, from eq. (5),

$$F_2(\zeta)=F_1(\zeta)+\frac{e^{q\zeta_2}}{q}(A_{2,0}+A_{2,1}\zeta_2)-\frac{A_{2,0}}{q}$$

$$+\frac{e^{p\zeta_2}}{p}(B_{2,0}+B_{2,1}\zeta_2)-\frac{B_{2,0}}{p}, \quad \ldots \quad (7a)$$

where,

$$\left.\begin{aligned}
A_{2,0}&=a_{2,0}+\frac{a_{2,1}}{q}=\frac{v_1}{c\beta^3}(2-\beta^2-2\beta), \quad (\textit{vide} \text{ eq. (22), Sec. II}),\\
B_{2,0}&=b_{2,0}+\frac{b_{2,1}}{p}=-\frac{v_1}{c\beta^3}(2-\beta^2+2\beta),\\
A_{2,1}&=a_{2,1}=\frac{v_1}{c\beta^3}\,2\beta q,\\
B_{2,1}&=b_{2,1}=\frac{v_1}{c\beta^3}\,2\beta p,
\end{aligned}\right\} \quad (7b)$$

which are obtained by putting $n=2$ and $r=0$ and 1, in eq. (3)

(*c*) $F(\zeta)$ during the interval $7l > \zeta - c\tau > 5l$

Putting $n=3$ in eq. (5), we have,

$$F_3(\zeta)=F_2(\zeta)+\frac{e^{q\zeta_3}}{q}(A_{3,0}+A_{3,1}\zeta_3+A_{3,2}\zeta_3^2)-\frac{A_{3,0}}{q}$$

$$+\frac{e^{p\zeta_3}}{p}(B_{3,0}+B_{3,1}\zeta_3+B_{3,2}\zeta_3^2)-\frac{B_{3,0}}{p}; \quad \ldots \quad (8a)$$

where

$$\left.\begin{aligned}
A_{3,0}&=a_{3,0}-\frac{a_{3,1}}{q}+2\frac{a_{3,2}}{q^2}=\frac{v_1}{c\beta^5}\{\beta^4+4\beta^3-2(1+\beta)(6\beta-2\beta^2-3)\},\\
&\qquad\qquad (cf\ \text{eq (23}b\text{), Sec. II}),\\
B_{3,0}&=b_{3,0}-\frac{b_{3,1}}{p}+2\frac{b_{3,2}}{p^2}=\frac{v_1}{c\beta^5}\{\beta^4+4\beta^3+2(1-\beta)(6\beta+2\beta^2+3)\},\\
A_{3,1}&=a_{3,1}-2\frac{a_{3,2}}{q}=\frac{v_1}{c\beta^5}(1-\beta)(3+2\beta)2\beta q,\\
B_{3,1}&=b_{3,1}-2\frac{b_{3,2}}{p}=\frac{v_1}{c\beta^5}(1+\beta)(3-2\beta)2\beta p,\\
A_{3,2}&=a_{3,2}=\frac{v_1}{c\beta^5}\frac{(2\beta q)^2}{2!},\quad B_{3,2}=b_{3,2}=\frac{v_1}{c\beta^5}\frac{(2\beta p)^2}{2!},
\end{aligned}\right\}\tag{8b}$$

which are obtained by putting $n=3$, $r=0, 1, 2$, in eq. (3).

(d) $F(\zeta)$ during the interval $9l>\zeta-c\tau>7l$

Putting $n=4$ in eq. (5), we have,

$$F_4(\zeta)=F_3(\zeta)+\frac{e^{q\zeta_4}}{q}\sum_{r=0}^{3}A_{4,r}\zeta_4^r-\frac{A_{4,0}}{q}+\frac{e^{p\zeta_4}}{p}\sum_{r=0}^{3}B_{3,r}\zeta_4^r-\frac{B_{3,0}}{p}\tag{9a}$$

where

$$\left.\begin{aligned}
A_{4,0}&=a_{4,0}-\frac{a_{4,1}}{q}+2\frac{a_{4,2}}{q^2}-6\frac{a_{4,3}}{q^3},\\
B_{4,0}&=b_{4,0}-\frac{b_{4,1}}{p}+2\frac{b_{4,2}}{p^2}-6\frac{b_{4,3}}{p^3},\\
A_{4,1}&=a_{4,1}-2\frac{a_{4,2}}{q}+6\frac{a_{4,3}}{q^2},\\
B_{4,1}&=b_{4,1}-2\frac{b_{4,2}}{p}+6\frac{b_{4,3}}{p^2},\\
A_{4,2}&=a_{4,2}-3\frac{a_{4,3}}{q},\quad B_{4,2}=b_{4,2}-3\frac{b_{4,3}}{p},\\
A_{4,3}&=a_{4,3},\qquad B_{4,3}=b_{4,3},
\end{aligned}\right\}\quad\cdots\tag{9b}$$

which are obtained by putting $n=4$ and $r=0, 1, 2, 3$ in eq. (3). The values of the right hand members are known from eq. (24b), Sec. II.

In a similar manner we can evaluate the function for any interval from the general expression given.

(ii) *Displacement at any point of the bar.*

The displacement at any point for any interval, during Hooke's period, can directly be obtained by the help of eq. (5), Sec. I and from $F(\zeta)$ during the interval. It is given by

$$w_n = F_{n-1}(ct-s) - F_n(ct+s) \qquad \dots \quad (10)$$

where F_{n-1} and F_n are fully known from eq. (5). But putting $n=1$ in eq. (10) we get, for the displacement during the interval $2l > c(t-\tau) > l$

$$w_1 = -F_1(ct+s), \qquad \text{since } F_0 = 0. \qquad \dots \quad (11)$$

Similarly, when $n=2$ we find that w, during $4l > c(t-\tau) > 2l$, is given by

$$w_2 = F_1(ct+s) - F_2(ct+s). \qquad \dots \quad (12)$$

In the like manner, we get the values for the interval higher than the above.

(iii) *Displacement at the struck point.*

This is obtained by putting $s=l$ in eq. (10), or we have

$$(w_n)_{s=l} = F_{n-1}(ct-l) - F_n(ct+l)$$

$$= (w_{n-1})_{s=l} - \Big[\frac{e^{q(ct_1 - \overline{n-1}\,2l)}}{q} \sum_{r=0}^{n-1} (A_{n,r} - A_{(n-1),r})(ct_1 - \overline{n-1}.2l)^r$$

$$- \frac{A_{n,0} - A_{(n-1),0}}{q} + \frac{e^{p(ct_1 - \overline{n-1}\,2l)}}{p} \Sigma (B_{n,r} - B_{(n-1),r})(ct_1 - \overline{n-1}.2l)^r$$

$$- \frac{B_{n,r} - B_{(n-1),r}}{p} \Big], \qquad \dots \quad (13)$$

where $t_1 = t - \tau$ $\qquad \dots \quad (15)$

By giving different values to n in eq. (13), we get

$$(w_1)_{s=l} = -F_1(ct+l) = -\frac{v}{c\beta}\left[\frac{e^{qct_1}-1}{q} - \frac{e^{pct_1}-1}{p}\right], \quad \dots \quad (15a)$$

$$(w_2)_{t=t}=\mathrm{F}_1(ct-l)+\mathrm{F}_2(ct+l)$$

$$=(w_1)_{t=t}+\frac{2v_1}{c\beta^3}\Bigg[\frac{e^{qct_2}}{q}(\beta^2+\beta-1-\beta qct_2)-\frac{\beta^2+\beta-1}{q}$$

$$-\frac{e^{pct_2}}{p}(\beta^2+\beta-1+\beta pct_2)+\frac{\beta^2+\beta-1}{p}\Bigg], \quad . \quad (15b)$$

where $t_2=t_1-2l$,

and

$$(w_3)_{t=t}=(w_2)_{t=t}-\Bigg[\frac{e^{qct_3}}{q}\sum_{r=0}^{2}(\mathrm{A}_{3,r}-\mathrm{A}_{2,r})(ct_3)^r-\frac{\mathrm{A}_{3,0}-\mathrm{A}_{2,0}}{q}$$

$$+\frac{e^{pct_3}}{p}\sum_{r=0}^{2}(\mathrm{B}_{3,r}-\mathrm{B}_{2,r})(ct_3)^r-\frac{\mathrm{B}_{3,0}-\mathrm{B}_{2,0}}{p}\Bigg], \quad \ldots \quad (15c)$$

where $t_3=t_2-2l$ and

$$\mathrm{A}_{3,0}-\mathrm{A}_{2,0}=\frac{2v_1}{c\beta^5}\,[\beta^4+3(1+\beta)(1-\beta)^2],$$

$$\mathrm{A}_{3,1}-\mathrm{A}_{2,1}=\frac{2v_1}{c\beta^5}\,[\beta^2-(1-\beta)(3+2\beta)]\beta q,$$

$$\mathrm{A}_{3,2}=\frac{2v_1}{c\beta^5}\,\beta^2q^2,$$

$$\mathrm{B}_{3,0}-\mathrm{B}_{2,0}=\frac{2v_1}{c\beta^5}\,[2\beta^2+3(1-\beta)(1+\beta)^2],$$

$$\mathrm{B}_{3,1}-\mathrm{B}_{2,1}=-\frac{2v_1}{c\beta^5}\,[\beta^2-(1+\beta)(3-2\beta)]\beta p,$$

$$\mathrm{B}_{3,2}=-\frac{2v_1}{c\beta^5}\,\beta^2p^2.$$

In a similar manner, we can find out the displacement of the struck point, at any interval. It should be noted that in order to get the complete displacement from the beginning of the impact we have to add w_0 the local statical compression produced during Hertz's period to the expression of the displacement during Hooke's period.

Pressure exerted by the load.

From eq (2) and (6a) Sec I, the pressure exerted by the load is given by

$$P = Ea[F'(ct-l)+F'(ct+l)] \qquad \ldots \quad (16)$$

which when written in our usual notations, the pressure for n_{th} interval is represented by

$$P_n = E_1 a[F'_n(\zeta)+F'_{n-1}(\zeta-2l)]$$

$$= P_{n-1} + E_1 a[\phi_n(\zeta_n)+\phi_{n-1}(\zeta_{n-1})] \qquad \ldots \quad (17a)$$

where we write ζ for $ct+l$ only, and ζ_n has the corresponding usual meaning given by eq (1), Sec II. By the help of eq (5), Sec. II

$$P_n = P_{n-1} + E_1 a[e^{q\zeta_n}\sum_{r=0}^{n-1}(a_{n,r}+a_{n-1,r})\zeta_n^r$$

$$+ e^{p\zeta_n}\sum_{r=0}^{n-1}(b_{n,r}+b_{n-1,r})\zeta_n^r] \qquad \ldots \quad (17b)$$

Now giving different values of n, we get the pressure for different intervals, as follows

$$P_1 = E_1 a[a_{1,0}e^{qct_1}+b_{1,0}\,e^{pct_1}] = \frac{\rho v_1 c}{\beta}\,(e^{qct_1} - e^{pct_1}); \qquad \ldots \quad (19)$$

$$P_2 = P_1 + E_1 a[e^{qct_2}\{(a_{2,0}+a_{1,0})+a_{2,1}ct_2\} + e^{pct_2}\{(b_{2,0}+b_{1,0})+b_{2,1}ct_2\}$$

$$= P_1 + \frac{2\rho v_1 c}{\beta^3}\,[e^{qct_2}(1+\beta qct_2) - e^{pct_2}(1-\beta ct_2)]; \qquad \ldots \quad (19)$$

$$P_3 = P_2 + E_1 a[e^{qct_3}\sum_{r=0}^{2}(a_{3,r}+a_{2,r})(ct_3)^r + e^{pct_3}\sum_{r=0}^{2}(b_{3,r}+b_{2,r})(ct_3)^r],$$

$$= P_2 + \frac{2\rho v_1 c}{\beta^5}\,[e^{qct_3}\{(3-2\beta^2)+(3+\beta-\beta^2)\beta qct_3+\beta^2 q^2 c^2 t_3^2\}$$

$$+ e^{pct_3}\{(3-2\beta^2)-(3-\beta-\beta^2)\beta pct_3-\beta^2 p^2 c^2 t_3^2\}]; \qquad \ldots \quad (20)$$

and

$$P_4 = P_3 + E_1 a \Big[e^{qct_4} \sum_{r=0}^{3} (a_{4,r} + a_{3,r})(ct_4)^r$$

$$+ e^{pct_4} \sum_{r=0}^{3} (b_{4,r} + b_{3,r})(ct_4)^r \Big], \quad \ldots \quad (21)$$

where $a_{3,r}$ $b_{3,r}$ and $a_{4,r}$ $b_{4,r}$ are given by eqs (23*b*) and (24*b*), Sec. II. In the similar manner we can find out P_5, P_6, etc.

So long we have considered the case where q and p, as given by eq (11), Sec I, are real But q and p are imaginary if the expression under the radical sign of eq. (11) Sec I be negative, *i e*, if

$$\frac{4E_2}{Mc^2} > \frac{E_2^2}{E_1^2 a^2}$$

$$\text{or} \qquad \frac{4a\rho}{M} > \frac{E_2}{E_1} \text{ as } c^2\rho = E_1 a \quad \ldots \quad (22)$$

This may very easily be realised if the load is very light.

Thus in the present case we may write

$$\left. \begin{aligned} q &= \mu + i\nu \\ p &= \mu - i\nu \end{aligned} \right\} \quad \ldots \quad (23)$$

where

$$\mu = -\frac{E_2}{2E_1 a} \text{ and } \nu = \tfrac{1}{2} \sqrt{\left(\frac{4E_2}{Mc^2} - \frac{E_2^2}{E_1^2 a^2} \right)}, \quad \ldots \quad (24a)$$

$$\text{so} \qquad \beta = \lambda(q-p) = i\nu\lambda. \quad \ldots \quad (24b)$$

With these modified values of q, p and β, all the expressions obtained before can be very easily rewritten We however give here only the transformations of eq (19) and (20)

$$P_1 = \frac{\rho v_1 c}{\lambda \nu} e^{\mu c t_1} \sin \nu c t_1, \quad \ldots \quad (25)$$

$$P_2 = P_1 + \frac{\rho v_1 c}{\lambda^2 \nu^2} \Big[\sqrt{(\mu^2 + \nu^2)}\, ct_2 \operatorname{Sin} \left(\nu c t_2 - \tan^{-1} \frac{\mu}{\nu}\right) - \frac{\sin \nu c t_2}{2\lambda\nu} \Big]. \quad (26)$$

Section IV

So far we have developed the theory in the case, when the bar is struck by an elastic load. In the present section, we shall show how the general solution may be reduced to the case of a hard load.

In this case $\xi=0, v_1 \longrightarrow v_0$ and $E_2=\infty$, hence

$$\beta=1, \lambda=0 \quad p=-\infty, q=-\frac{1}{ml} \qquad \text{.. (1)}$$

where $m=\frac{M}{\rho l}$. Moreover τ is zero, as the impact does not begin, following Hertz's law.

Substituting the above values in eq. (5), Sec. II, we get,

$$\phi_n(\zeta_n)=e^{q\zeta_n}\sum_{r=0}^{n-1} a_{n,r}\zeta_n^r, \qquad \text{... (2)}$$

and from eqs. (7), (17), (18) and (19), we have,

$$\left.\begin{aligned}
&a_{n,0}-a_{(n-1),0}=0 \text{ or } a_{n,0}=a_{1,0}=\frac{v_0}{c},\\
&a_{n,(r+1)}=a_{(n-1),(r+1)}+\frac{2q}{r+1}a_{(n-1),r},\\
&a_{n,(n-1)}=\frac{(2q)^{n-1}}{(n-1)!}\,a_{1,0},\\
&a_{n,(n-2)}=(2q)^{n-2}\,\frac{(n-1)}{(n-2)!}\,a_{1,0},
\end{aligned}\right\} \qquad \text{... (3)}$$

which lead to, for suitable values of n and r,

$$\left.\begin{aligned}
&a_{2,0}=\frac{v_0}{c},\ a_{2,1}=\frac{v_0}{c}(2q);\\
&a_{3,0}=\frac{v_0}{c},\ a_{3,1}=\frac{v_0}{c}2(2q),\ a_{3,2}=\frac{v_0}{c}\frac{(2q)^2}{2!},\\
&a_{4,0}=\frac{v_0}{c},\ a_{4,1}=\frac{v_0}{c}3(2q),\ a_{4,2}=\frac{v_0}{c}3\frac{(2q)^2}{2!},\ a_{4,3}=\frac{v_0}{c}\frac{(2q)^3}{3!},\\
&a_{5,0}=\frac{v_0}{c},\ a_{5,1}=\frac{v_0}{c}4(2q),\ a_{5,2}=\frac{v_0}{c}6\frac{(2q)^2}{2!},\ a_{5,3}=\frac{v_0}{c}4\frac{(2q)^3}{3!},\\
&\qquad\qquad a_{5,4}=\frac{v_0}{c}\frac{(2q)^4}{4!};
\end{aligned}\right\} \qquad \text{(4)}$$

where $q=-\frac{1}{ml}$ as stated before. These lead to identical expressions of $F'(\zeta)$ for different intervals as given by Love,* in his treatise on Elasticity. It should be noted that Love has given the expressions only up to the third intervals $7l>\zeta>5l$, owing to some algebraic difficulties. But the present method is perfectly straightforward and can be used to evaluate $F'(\zeta)$ for any interval.

In the present case eq (5), Sec. III, reduces to

$$F_n(\zeta)=F_{n-1}(\zeta)+\frac{e^{q\zeta_n}}{q}\sum_{r=0}^{(n-1)}A_{n,r}\zeta_n^r-\frac{A_{n,0}}{q} \quad \cdots \quad (5)$$

where

$$A_{n,r}=\Big[a_{n,r}-{}^{(r+1)}P_1.\frac{a_{n,(r+1)}}{q}+{}^{(r+2)}P_2.\frac{a_{n,(r+2)}}{q^2}-\cdots\cdots\cdots$$

$$\cdots+(-)^{n-1-r}\,{}^{(n-1)}P_{(n-1-r)}.\frac{a_{n,(n-1)}}{q^{n-1-r}}\Big].$$

This gives, for different values of n,

$$\left.\begin{aligned}
F_1(\zeta)&=\frac{v_0}{qc}(e^{q\zeta_1}-1),\\
F_2(\zeta)&=F_1(\zeta)-\frac{v_0}{qc}\Big[e^{q\zeta_2}(1-2q\zeta_2)-1\Big],\\
F_3(\zeta)&=F_2(\zeta)+\frac{v_0}{qc}\Big[e^{q\zeta_3}\Big\{1+\frac{(2q\zeta_3)^2}{2!}\Big\}-1\Big],\\
F_4(\zeta)&=F_3(\zeta)-\frac{v_0}{cq}\Big[e^{q\zeta_4}\Big\{1-(2q\zeta_4)-\frac{(2q\zeta_4)^2}{2!}-\frac{(2q\zeta_4)^3}{3!}\Big\}-1\Big],\\
F_5(\zeta)&=F_4(\zeta)+\frac{v}{qc}\Big[e^{q\zeta_5}\Big\{1+(2q\zeta_5)^2+\frac{2}{3!}(2q\zeta_5)^3\\
&\qquad+\frac{1}{4!}(2q\zeta_5)^4\Big\}-1\Big].
\end{aligned}\right\}\quad(6)$$

* Love :—The Mathematical Theory of Elasticity, 4th edition art 281, pp. 431 441.

From eq (6), we can easily get the expression for the displacement at the struck point by putting $(ct-s)$ and $(ct+s)$ for ζ as required by the eq (5), Sec I, and finally putting $s=l$. These are given by,

$$(w_1)_{s=l}=-\frac{v_0}{qc}(e^{qct}-1),$$

$$(w_2)_{s=l}=(w_1)_{s=l}+\frac{2v_0}{qc}\left[e^{q(ct-2l)}\{1-q(ct-2l)\}-1\right],$$

$$(w_3)_{s=l}=(w_2)_{s=l}-\frac{2v_0}{qc}\left[e^{q(ct-4l)}\{1-q(ct-4l)+q^2(ct-4l)^2\}-1\right]$$

$$(w_4)_{s=l}=(w_3)_{s=l}+\frac{2v_0}{qc}\left[e^{q(ct-6l)}\{1-q(ct-6l)-\tfrac{2}{3}q^3(ct-6l)^3\}-1\right],$$

$$(w_5)_{s=l}=(w_4)_{s=l}-\frac{2v_0}{qc}\left[e^{q(ct-8l)}\{1-q(ct-8l)+q^2(ct-8l)^2\right.$$

$$\left.+\frac{4}{3!}q^3(ct-8l)^3+\tfrac{1}{4}q^4(ct-8l)^4\}-1\right], \quad \ldots \quad (7)$$

and the pressures exerted by the hard lead at different intervals may be obtained by differentiating $w_{s=l}$ twice with respect to time as $P=M\ddot{w}_{s=l}$ or they may be directly obtained from $F'(\zeta)$. The general expression for the pressure is given by

$$P_n=P_{n-1}+E_1ae^{q(ct-\overline{n-1}\,2l)}\sum_{r=0}^{n-1}(a_{n,r}+a_{(n-1),r})(ct-\overline{n-1}\,2l)^r. \quad . . (8)$$

By the help of eqs (4) and (9), we get,

$$P_1=\rho v_0ce^{qct},$$

$$P_2=P_1+2\rho v_0ce^{q(ct-2l)}\{1+q(ct-2l)\},$$

$$P_3=P_2+2\rho v_0ce^{q(ct-4l)}\{1+3q(ct-4l)+q^2(ct-4l)^2\},$$

$$P_4=P_3+2\rho v_0ce^{q(ct-6l)}\{1+5q(ct-6l)+4q^2(ct-6l)^2$$

$$+\frac{4}{3!}q^3(ct-6l)^3\},$$

$$P_5 = P_4 + 2\rho v_0 c e^{q(ct-8l)}\{1 + 7q(ct-8l) + 9q^2(ct-8l)^2$$

$$+ \frac{20}{3!} q^3 (ct-8l)^3 + \frac{8}{4!} q^4 (ct-8l)^4\}, \qquad \ldots \quad (9)$$

and so on. In a similar manner we can determine the pressure at any interval. Thus it is seen *unlike the elastic load* that the pressure in the case of the hard load increases by a sudden jump of magnitude $2\rho v_0 c$ except at the beginning where it is $\rho v_0 c$ only.

The deduction in the case of the hard load which we have undertaken here plays an important role in developing Kaufmann's theory of the vibration of the Pianoforte string. The complete discussion of this point is given in one of our papers * on the subject.

Section V.

Duration of Contact.

The expressions for the pressure exerted by the load which we have derived above, is a function of time, and is taken to be measured from the beginning of the Hooke's period, that is $t = \tau$, where t represents time, measured from the beginning of the impact and τ, the Hertz's period at the beginning. In the case, when Hertz's period at the beginning is absent, i.e., $\tau = 0$ the duration of contact Φ is defined as the positive root of the pressure equation $P_n = 0$. But when $\tau \neq 0$, the positive root of the pressure equations will represent the sum of the Hertz's period at the beginning and the Hooke's period. But this is not the total duration of contact Φ, but is less by the amount of the Hertz's period at the end which is also taken to be equal to τ. So substituting $\Phi - \tau$ for t in the pressure equation and solving for the positive values of Φ we get the required duration of contact. It may be noted that at higher intervals Φ may have multiple number of positive values. This will explain the multiple contact during impact.

From eq. (15), Sec. III, we find $P_1 = 0$ has got no real root except 0 and ∞; so the impact does not terminate during the first interval. But in the case of light and soft load, eq. (18), Sec. III, is transformed into

* Kar—Ghosh :—Zeit. f. Phy. Vol. 61, pp. 526-537 (1930).

what is given by eq (26) In this case the desired root, by putting $\Phi-\tau$ for t, is given by

$$\Phi = \frac{\pi}{\nu c} + 2\tau,$$

where ν is given by eq (24*a*), Sec III.

Again substituting $\Phi-\tau$ for t in P_4 given by eq (19), Sec III and equating to zero, we get

$$\Phi = 2\tau + \frac{2l}{c} + \frac{(\beta e^{2pl}+2)\ \mathrm{Exp}[(p-q)c(\Phi-2\tau-\frac{2l}{c})] - (\beta e^{2ql}+2)}{2\beta c[q+p\,\mathrm{Exp}\{(p-q)c(\Phi-2\tau-\frac{2l}{c})\}]} \qquad (2)$$

Here the last expression on the right hand side involves Φ in the exponential, so it is difficult to make an exact evaluation of it An approximate value may however be obtained in the following way;

As the pressure terminates during the second interval, so $(\Phi-2\tau-\frac{2l}{c})$ must lie between 0 and $\frac{2l}{c}$ for all admissible values of q, p and c Again Exp $(p-q)c(\Phi-2\tau-\frac{2l}{c})$ lies within the range 1 and 0 when $(\Phi-2\tau-\frac{2l}{c})$ varies from 0 to ∞ Therefore the value of the exponential undergoes only a small change when $(\Phi-2\tau-\frac{2l}{c})$ changes from 0 to $\frac{2l}{c}$ Hence we may put the mean value $\frac{l}{c}$ for $(\Phi-2\tau-\frac{2l}{c})$ in the exponential, without introducing any serious error in the evaluation of Φ. Thus eq. (2) becomes

$$\Phi = 2\tau + \frac{2l}{c} + \frac{(\beta e^{2pl}+1)e^{(p-q)l} - (\beta e^{2ql}+2)}{2\beta c\{q+pe^{(p-q)l}\}}, \qquad \dots \quad (3)$$

which is the duration of contact when pressure terminates during the second interval.

In the same way we may proceed to obtain Φ up to the interval $10l > c(t-\tau) > 8l$, beyond which algebraic solution is not possible, so graphical method should be adopted in those cases.

In the case of the light and soft load when $[q, p] = \mu \pm i\nu$, Φ will be obtained from eq (27), Sec III, in the same manner But here, too, algebraic solution fails Further it is unwise to reduce eq (3) to the case of the light and soft load, owing to the approximation introduced.

In the case of the hard load where $q=-\frac{1}{ml}$ (*vide* eq. (1), Sec. III) the pressure does not terminate during the interval $2l>ct>0$ which is evident from eq. (9a), Sec. IV.

If the pressure terminates during the interval $4l>ct>2l$ the duration of contact is obtained by equating eq. (9b) to zero and substituting Φ for t. We have,

$$\Phi=T\left[1+\frac{m}{4}\left(2+e^{-\frac{2}{m}}\right)\right], \qquad \ldots \quad (4a)$$

where

$$T=\frac{2l}{c}, \qquad \ldots \quad (4b)$$

provided the mass ratio m does not exceed the value $m=1\cdot7$, being the root of the equation,

$$\frac{m}{4}\left(2+e^{-\frac{2}{m}}\right)=1$$

Thus the pressure terminates during this interval so long as $m \ngtr 1.7$. The eq. (4) can also be readily obtained from (3) with the approximations required for the hard load. The above values of Φ and m are also given by Love (*loc. cit.*)

In order to obtain the duration of contact where pressure terminates during the interval $6l>ct>4l$, we get from eq. (9c), Sec. IV, in the same manner as before

$$(\Phi-2T)^2-(\gamma_2)_1T(\Phi-2T)+(\gamma_2)_2T^2=0, \qquad \ldots \quad (6)$$

where

$$\left.\begin{aligned}(\gamma_2)_1&=\frac{m}{2}\left(3+e^{-\frac{2}{m}}\right),\\ (\gamma_2)_2&=\frac{m^2}{4}\left[1+\frac{e^{-\frac{4}{m}}}{2}+e^{-\frac{2}{m}}\left(1-\frac{2}{m}\right)\right],\end{aligned}\right\} \qquad \ldots \quad (7)$$

provided the maximum mass ratio is given by

$$1-(\gamma_2)_1+(\gamma_2)_2=0, \qquad \ldots \quad (8)$$

which has a root $m=4.14$ approximately. So the pressure terminates during $6l>ct>4l$ so long as $m \ngtr 4.14$ and the duration of contact which is obtained on solving (6) is given by

$$\Phi=2T+T\frac{m}{4}\left[\left(3+e^{-\frac{2}{m}}\right)\pm\left\{\left(3+e^{-\frac{2}{m}}\right)^2-4\left(1+\frac{e^{-\frac{4}{m}}}{2}+e^{-\frac{2}{m}}\left(1-\frac{2}{m}\right)\right)\right\}^{\frac{1}{2}}\right] \qquad \ldots \quad (9)$$

it may be noted that as double sign occurs in the value of Φ_2 we take only the sign that makes $3T>\Phi>2T$ for particular value of m lying between 1·7 and 4·14

Similarly for the interval $8l>ct>6l$, we get, by equating P_4 given by eq. (9d), Sec IV to zero, the duration of contact Φ as the positive root of the equation

$$(\Phi-3T)^3-(\gamma_3)_1T(\Phi-3T)^2+(\gamma_3)_2T^2(\Phi-3T)-(\gamma_3)_3T^3=0, \tag{10}$$

where

$$\left.\begin{aligned}
(\gamma_3)_1&=\frac{3m}{4}\left(4+e^{-\frac{2}{m}}\right),\\
(\gamma_3)_2&=\frac{3m^2}{8}\left[5+e^{-\frac{4}{m}}+e^{-\frac{2}{m}}\left(3-\frac{4}{m}\right)\right],\\
(\gamma_3)_3&=\frac{3m^3}{16}\left[1+\frac{e^{-\frac{6}{m}}}{2}+e^{-\frac{4}{m}}\left(1-\frac{4}{m}\right)+e^{-\frac{2}{m}}\left(1-\frac{6}{m}+\frac{4}{m^2}\right)\right],
\end{aligned}\right\} \tag{11}$$

provided the maximum mass ratio is given by

$$1-(\gamma_3)_1+(\gamma_3)_2-(\gamma_3)_3=0, \quad \dots \tag{12}$$

which has a root $m=7·3$, so the pressure terminates during this interval, so long $m \not> 7·3$ and $\not< 4·14$.

During the interval $10l>ct>8l$ we get, by equating P_5 to zero, the duration of contact as the root of the equation

$$(\Phi-4T)^4-(\gamma_4)_1T(\Phi-4T)^3+(\gamma_4)_2T^2(\Phi-4T)^2$$
$$-(\gamma_4)_3T^3(\Phi-4T)+(\gamma_4)_4T^4=0 \quad \dots \tag{13}$$

where

$$\left.\begin{aligned}
(\gamma_4)_1&=m(5+e^{-\frac{2}{m}}),\\
(\gamma_4)_2&=\frac{3m^2}{4}\left[9+e^{-\frac{4}{m}}+4e^{-\frac{2}{m}}\left(1-\frac{1}{m}\right)\right],\\
(\gamma_4)_3&=\frac{3m^3}{8}\left[7+e^{-\frac{6}{m}}+e^{-\frac{4}{m}}\left(3-\frac{8}{m}\right)+e^{-\frac{2}{m}}\left(5-\frac{16}{m}+\frac{8}{m^2}\right)\right]\\
(\gamma_4)_4&=\frac{3m^4}{16}\left[1+\frac{e^{-\frac{8}{m}}}{2}+e^{-\frac{6}{m}}\left(1-\frac{6}{m}\right)+e^{-\frac{4}{m}}\left(1-\frac{12}{m}+\frac{16}{m^2}\right)\right.\\
&\qquad\left.+e^{-\frac{2}{m}}\left(1-\frac{10}{m}+\frac{16}{m^2}-\frac{16}{3m^3}\right)\right];
\end{aligned}\right\} \tag{14}$$

provided the maximum mass ratio is given by

$$1-(\gamma_4)_1+(\gamma_4)_2-(\gamma_4)_3+(\gamma_4)_4=0, \quad \dots \tag{15}$$

which has a root $m=10\cdot 4$, so the pressure terminates during the interval $10l>ct>8l$ if the mass ratio lies between 7 3 and 10 4

The general solution of the eqs (10) and (13) are well known in the theory of equation But the pressure for an interval higher than $10l>ct>8l$ will lead to similar equations of order higher than four. So in that case algebraic solution fails and the numerical solution is to be adopted

Summary

The dynamical theory of collision of a hard load with the free end of a bar whose other end is fixed was developed by Bousinesque. This theory is extended to the case of an elastic load obeying Hooke's law of compression. In doing so it is assumed that the collision begins following Hertz's law of impact, until pressure exerted by the load attains a finite value, beyond which the compression of the load follows Hooke's law, and waves are generated in the bar from the struck end After a time, Hooke's law is over and the pressure falls to the same finite value. From this value pressure falls to zero following Hertz's law again, till there is no more contact The calculation for Hertz's period is not developed in this paper. The detailed calculations for Hooke's period are given in different sections. In Sec. I, the Equation Prometrice which has undergone modification due to the introduction of elasticity of the load is solved successively for different intervals in a much simpler manner, by adopting the symbolic representation of the differential operator. In Sec. II, we have developed a general method of solving the problem. This eliminates the trouble of successive integrations and allows us to know the complete solution, from the knowledge of the same at the beginning of the interval In Sec III, the general expression for the displacement, pressure, etc., are given, from which, special cases for any interval are easily deduced. In Sec IV the generalised treatment in the case of the hard load is given, from which the functions for different intervals are obtained as special cases In Sec V expressions for the duration of contact for different cases are obtained and other related questions are discussed

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Zahlentheoretische Untersuchungen und Resultate

BY

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(Communicated by the Secretary)

In nachfolgenden Zeilen sollen gewisse zahlentheoretische Resultate auf die elementarste Weise gewonnen werden.

I. (α) Nach der Formel $2(1+x+x^2)^n=(x^2-1)^n+(1+2x)^n+(x^2+2x)^n$ für $n=2$ und 4 bekommt man ganzzahlige Lösungen der Gleichung $2\,A^n=B^n+C^n+(B+C)^n$ für $n=2$ und 4. Setzt man zum Exempel $x=3$, so bekommt man $2\cdot 13^n=8^n+7^n+15^n$ für $n=2$ und 4. Nun ist aber die Identität $2\cdot A^n=B^n+C^n+(B+C)^n$ für $n=2$ und 4 der Gleichung $2\,(A^2)^n=(B^2)^n+(C^2)^n+[(B+C)^2]^n$ für $n=1$ und 2 gleich. Diese letzte Gleichung kann man auch in der Form schreiben: $0^n+(A^2)^n+(A^2)^n=(B^2)^n+(C^2)^n+[(B+C)^2]^n$ für $n=1$ und 2. Besteht die Relation $J_1^n+J_2^n+J_3^n=K_1^n+K_2^n+K_3^n$ für $n=1$ und 2, dann gilt nach einem allgemeinen Theorem zugleich $(J_1\pm s)^n+(J_2\pm s)^n+(J_3\pm s)^n=(K_1\pm s)^n+(K_2\pm s)^n+(K_3\pm s)^n$ für $n=1$ und 2. Setzt man nun $s=A^2$ und nimmt bei $0^n+(A^2)^n=(B^2)^n+(C^2)^n+[(B+C)^2]^n$ für $n=1$ und 2 die subtraktive Veränderung vor, dann bekommt man die Relation $(-A^2)^n=(-BC-B^2)^n+(-BC-C^2)^n+(BC)^n$ für $n=1$ und 2 oder (mit -1 multipliziert): $(A^2)^n=(BC+B^2)^n+(BC+C^2)^n+(-BC)^n$ für $n=1$ und 2.

(β) Aus (α) ergibt sich der allgemeine Satz: "Gilt die Relation $2\cdot A^n=B^n+C^n+(B+C)^n$ für $n=2$ und 4, dann besteht auch die Relation $(A^2)^n=(BC+B^2)^n+(BC+C^2)^n+(-BC)^n$ für $n=1$ und 2. Dabei ist $(BC+B^2)+(BC+C^2)=(B+C)^2$ und $(BC+C^2)+(-BC)=C^2$."

Exempel. $2\cdot 19^n=16^n+5^n+(16+5)^n$ für $n=2$ und 4, also $19^2=(16\cdot 5+16^2)+(16\cdot 5+5^2)+(-16\cdot 5)=361$ und $336^2+105^2+(-80)^2=(19^2)^2=19^4$; ferner ist $336+105=441=(16+5)^2$ und $105+(-80)=25=5^2$.

(γ) Besteht die Relation $0^n+(A^2)^n+(A^2)^n=(B^2)^n+(C^2)^n+[(B+C)^2]^n$ für $n=1$ und 2 und setzt man $s=\dfrac{A^2+C^2}{2}$, wobei $C<B$ ist, so ergibt

$(O^2-s)^n+(A^2-s)^n+(A^2-s)^n=(B^2-s)^n+(C^2-s)^n+[(B+C)^2-s)]^n$ für $n=1$ und 2 Identitäten von der Form $D^n+(-E)^n+(-E)^n=E^n+F^n+(-G)^n$ für $n=1$ und 2, wobei $D=s$ ist und wobei $E+F=BC+C^2$ und $E=\frac{BC+B^2}{2}$ ist Aus $D^n+(-E)^n+(-E)^n=E^n+F^n+(-G)^n$ für $n=1$ und 2 folgt, $D^2+(-E)^2=F^2+(-G)^2$, wobei $(B+C)^2=3E+F$ ist und $D+E=A^2$ ist.

Exempel $0^n+(7^2)^n+(7^2)^n=(5^2)^n+(3^2)^n+[(5+3)^2]^n$ für $n=1$ und 2, $s=\frac{7^2+3^2}{2}=29$ ergibt $29^n+(-20)^n+(-20)^n=20^n+4^n+(-35)^n$ für $n=1$ und 2; $29^2+(-20)^2=4^2+(-35)^2$, $29+20=7^2$, $3.\ 20+4=(5+3)^2$.

(δ) Die Untersuchung zeigt, dass, wenn man bei 2. $(A^2)^n=(B^2)^n+(C^2)^n+[(B+C)^2]^n$ für $n=1$ und 2 für $s=\frac{A^2+B^2}{2}$ setzt, sich Identitäten von der Form $(-D)^n+E^n+E^n=(-F)^n+(-E)^n+G^n$ für $n=1$ und 2 ergeben, wobei $(-D)=-s$ ist. Daraus folgt: $D^2+E^2=F^2+G^2$; ferner $D+E=A^2$ und $F+3.\ E=G+D=(B+C)^2$. Setzt man $s=\frac{A^2+(B+C)^2}{2}$, so ergibt 2 $(A^2-s)^n=(B^2-s)^n+(C^2-s)^n+[(B+C)-s]^n$ für $n=1$ und 2 Identitäten von der Form $(-D)^n+(-E)^n+(-E)^n=(-F)^n+(-G)^n+E^n$ für $n=1$ und 2; und folglich ist $D^2+E^2=F^2+G^2$, wobei $D=s$ und $D+E=(B+C)^2$ ist

II. (α) Nach der Formel $(f-2g)^n+(4f-g)^n+(3g-5f)^n=(4f-3g)^n+(2g-5f)^n+(f+g)^n$ für $n=1$, 2 und 4 bekommt man ganzzahlige Lösungen für $H_1^n+H_2^n+H_3^n=L_1^n+L_2^n+L_3^n$ für $n=1$, 2 und 4 (wobei auch negative Glieder auftreten); berücksichtigt man die negativen Vorzeichen nicht, so hat man Lösungen für $H_1^n+H_2^n+H_3^n=L_1^n+L_2^n+L_3^n$ für $n=2$ und 4 Es ergibt zum Exempel $f=3$, $g=1$ das numerische Beispiel $1^n+11^n+(-12)^n=9^n+(-13)^n+4^n$ für $n=1$, 2 und 4, also $1^n+11^n+12^n=9^n+13^n+4^n$ für $n=2$ und 4 Es bedarf keines Kommentars, dass mit der angegebenen Formel zugleich

$$\left.\begin{aligned} X^2+Y^2+(X+Y)^2&=Z^2+U^2+(Z+U)^2 \\ X^4+Y^4+(X+Y)^4&=Z^4+U^4+(Z+U)^4 \end{aligned}\right\}\text{ gelöst ist.}$$

Wir setzen $X+Y=P$ und $Z+U=Q$, sodass also $X^4+Y^4+P^4=Z^4+U^4+Q^4$ ist. Dann muss nach einem bekannten Satze (wenn $Y>X$ und $U>Z$ ist)

$$X^4+Y^4+P^4=Z^4+U^4+Q^4=2(X^2+YP)^2=2(Z^2+UQ)^2$$

$$=2(Y^2+XP)^2=2(U^2+ZQ)^2=2[(XY)^2+(XP)^2+(YP)^2]$$

$$=2[(ZU)^2+(ZQ)^2+(UQ)^2] \text{ sein.}$$

(β) Das numerische Beispiel $9^n+4^n+13^n=1^n+12^n+11^n$ für $n=2$ und 4 bekommen wir auch, wenn wir bei $(4p^4+2p^2q^2-3q^4)^n+(2pq)^{2n}+(4p^4-2p^2q^2-3q^4)^n=(2p^2)^{2n}+(3q^4)^n+(4p^4-3q^4)^n$ für $n=2$ und 4 setzen $p=1$, $q=2$

Diese Formel gibt also immer Lösungen für das System

$$\begin{cases} R^2+S^4+T^2=U^4+V^2+W^2 \\ R^4+S^8+T^4=U^8+V^4+W^4. \end{cases}$$

(γ) Es soll nun bei dem Problem unter (α) der Bedingung genügt sein, dass $X=A^2$ und $Y=B^2$ und dass $A^2+B^2=C^2$ ist, sodass also A, B und C pythagoraische Zahlen bilden. Wir haben dann das System

$$\begin{cases} (A^2)^2+(B^2)^2+(A^2+B^2)^2=Z^2+U^2+(Z+U)^2 \\ (A^2)^4+(B^2)^4+(A^2+B^2)^4=Z^4+U^4+(Z+U)^4 \end{cases} \quad \text{oder}$$

$$\begin{cases} A^4+B^4+C^4=Z^2+U^2+(Z+U)^2 \\ A^8+B^8+C^8=Z^4+U^4+(Z+U)^4. \end{cases}$$

Nun ist $(A^2)^2+(B^2)^2+(A^2+B^2)^2=Z^2+U^2+(Z+U)^2$ auch so darstellbar: (1) $2(A^4+A^2B^2+B^4)=2(Z^2+ZU+U^2)$, sodass also zunächst zu lösen wäre:

$$A^4+A^2B^2+B^4=Z^2+ZU+U^2.$$

Setzen wir nun — m und n als teilerfremd und $m>n$ vorausgesetzt — $A=m^2-n^2$ und $B=2mn$, so geht Gleichung (1) über in (2) $m^8+14m^4n^4+n^8=Z^2+ZU+U^2$. Diese Gleichung wird identisch befriedigt durch $Z=m^4-2m^3n-6m^2n^2+2mn^3+n^4$; $U=4m^3n-4mn^3$. Man bekommt also unendlich viele, aber *nicht alle* Lösungen von Gleichung (1), wenn man setzt:

$$(3) \quad Z=m^4-2m^3n-6m^2n^2+2mn^3+n^4;\ U=4m^3n-4mn^3,$$

$$A=m^2-n^2;\ B=2mn.$$

Exempel: $m=2$, $n=1$ ergibt $A=3$, $B=4$, $Z=-19$, $U=24$. Da aber wegen der Identität $x^2-xy+y^2=(x-y)^2+(x-y)y+y^2$ die Relation $(-19)^2+(-19)\,24+24^2=5^2+5.19+19^2$ besteht, so bekommen wir,

$$\left.\begin{array}{l} 3^4+4^4+5^4=5^2+19^2+24^2 \\ 3^8+4^8+5^8=5^4+19^4+24^4 \end{array}\right\}$$

Es ergibt sich die algebraische Identität

$$3^8+4^8+5^8=5^4+19^4+(5+19)^4=2[(3.4)^4+(3.5)^4+(4.5)^4]$$

$$=2[(3^2)^2+(4^2.5^2)]^2=2[5^2+19.24]^2=2.481^2.$$

Andere Exempel:

$$\left.\begin{array}{l} 5^4+12^4+13^4=59^2+120^2+(59+120)^2 \\ 5^8+12^8+13^8=59^4+120^4+(58+120)^4 \end{array}\right\}$$

Setzt man $m=4$, $n=1$, so bekommt man die Lösung

$$\left.\begin{array}{l} 8^4+15^4+17^4=41^2+240^2+(41+240)^2 \\ 8^4+15^8+17^8=41^4+240^4+(41+240)^4 \end{array}\right\}$$

III. Als "Curiosa" seien folgende singuläre Resultate angeführt:

Zu IIα): $3^2+19^2+(3+19)^2=6^2+17^2+23^2=10^2+15^2+23^2$

$$3^4+19^4+(3+19)^4=6^4+17^4+23^4$$

$$3^6+19^6+(3+19)^6=10^6+15^6+23^6$$

Zu IIγ):

$$21^4+28^4+35^4=245^2+931^2+(245+931)^2=21^2+1064^2+(21+1064)^2$$

$$21^8+28^8+35^8=245^4+931^4+(245+931)^4=21^4+1064^4+(21+1064)^4.$$

Ich kenne noch keine allgemeine Lösung der Identität

$$\left\{\begin{array}{l} A^4+B^4+C^4=Z^2+U^2+(Z+U)^2=D^2+E^2+(D+E)^2 \\ A^8+B^8+C^8=Z^4+U^4+(Z+U)^4=D^4+E^4+(D+E)^4, \end{array}\right.$$

doch scheint es nicht schwierig zu sein, eine allgemeine Lösungsmethode zu finden.

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3

Gewisse Fragen der Zahlentheorie

BY

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Frage: Wie heisst die allgemeine ganzzahlige Lösung der Identität

$$\left.\begin{aligned} P_1+P_2+P_3=Q_1+Q_2+Q_3=R_1+R_2+R_3 \\ P_1^3+P_2^3+P_3^3=Q_1^3+Q_2^3+Q_3^3=R_1^3+R_2^3+R_3^3 \end{aligned}\right\},$$

wenn nur *positive* Zahlen verwendet werden sollen?

Exempel bei Verwendung von positiven und negativen Zahlen:

I $$\left.\begin{aligned} 25+25-1=1+19+29=9+9+31, \\ 25^3+25^3-1^3=1^3+19^3+29^3=9^3+9^3+31^3 \end{aligned}\right\}, \text{ also}$$

$$5^6+5^6-1^6=1^6+19^3+29^3=3^6+3^6+31^3.$$

II $$\left.\begin{aligned} 81+81-101=16+16+29=25+25+11 \\ 81^3+81^3-101^3=16^3+16^3+29^3=25^3+25^3+11^3 \end{aligned}\right\}, \text{ also}$$

$$3^{12}+3^{12}-101^3=2^{12}+2^{12}+29^3=5^6+5^6+11^3.$$

III. $$\left.\begin{aligned} 100+100-4=4+76+116=36+36+124 \\ 100^3+100^3-4^3=4^3+76^3+116^3=36^3+36^3+124^3 \end{aligned}\right\}, \text{ also}$$

$$10^6+10^6-2^6=2^6+76^3+116^3=6^6+6^6+124^3.$$

Es sie A_1, A_2, A_3, A_4, A_5, eins arithmetische Reihe, welche aus ganzzahligsn Gliedern bsstsht, wovon nur 1 Glied keine ganze Quadratzahl ist. (*Exempel* · 7^2, 13^2, 17^2, 409, 23^2) Nach welcher Methode findet man solchs arithmetischs Reihen? Ist die arithmetische Progression A_1, A_2, A_3, A_4, A_5, A_6 in ganzen Zahlen moglich, wenn *nur* 1 *glied* keine ganzs Quadratzahl ist?

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4

On a Few Algebraic Identities

BY

S. C. Chakrabarti

(*Calcutta*)

In a previous communication* it was shown that the series of operations necessary for effecting the resolution of a factorable determinant, may be replaced by a single operation and that when this operation is performed on the same determinant, some algebraic identities are generally obtained. Several identities obtained in this manner have already been published and a few more are given in the present paper. In proving the identities given here the following two theorems are of great use :—

(*i*) 'If 1, a, a^2, a^3,...are used as the successive multipliers, the first element of the *rth* order of differences obtained from the series $u_0, u_1, u_2, u_3,...$is

$$\sum_{x=0}^{r} (-)^x \, u_x \, {}^rS_x \qquad \ldots \quad (1)$$

where ${}^rS_x$ denotes the sum of the products of r factors 1, a, a^2, a^3... a^{r-1} taken x at a time;

$${}^rS_x = 0, \text{ if } x \text{ is negative or } > r;$$

$${}^rS_x = 1, \text{ if } x = 0.\text{'}$$

(*ii*)

$${}^rS_x = \frac{\left[\begin{smallmatrix} r \\ 1 \end{smallmatrix}\right]}{\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} r-x \\ 1 \end{smallmatrix}\right]} \, {}^xS_x \qquad \ldots \quad (2)$$

* Chakrabarti, S. C. 'On the Evaluation of some Factorable Continuants.' *Bul. Cal. Math. Soc.*, Vol. 13, (1922-23), pp. 71-84 and Vol. 14, (1923-24), pp. 91-106.

where

$\left[{n \atop c}\right]=(a^n-1)(a^{n-1}-1)(a^{n-2}-1)\ldots(a^c-1)$, $n-c$ is a positive integer;

$\left[{n \atop c}\right]=a^n-1$, if $n=c$;

$\left[{n \atop c}\right]=1$, if $n-c$ is a negative integer;

and $\left[{n \atop c}\right]=0$, if n is a positive integer and c is 0 or negative.

1. Denote

$\left[{n \atop c}\right]_2=(a^n-1)(a^{n-2}-1)(a^{n-4}-1)\ldots(a^c-1)$,

n and c are both odd or even integers,

and $\left[{n \atop c}\right]_2=1$, if $c>n$.

Then

$$(i)\quad \sum_{x=0}^{k}(-)^x\left[{2k-1-2x \atop 1}\right]_2 {}^{2k}S_{2x}=(-)^k, \qquad (3)$$

$$\text{and}\quad (ii)\quad \sum_{x=0}^{k}(-)^x\left[{2k-1-2x \atop 1}\right]_2 {}^{2k+1}S_{1+2x}=(-)^k. \qquad (4)$$

Proof:—Let the left-sides of the above theorems be denoted respectively by C_k and D_k.

Then it can be shown by (1) that C_{k+1} is the first element of the $2k+2$th order of differences obtained from the series

$$\left[{2k+1 \atop 1}\right]_2,\ 0,\ -\left[{2k-1 \atop 1}\right]_2,\ 0,\ \left[{2k-3 \atop 1}\right]_2,\ 0,\ldots$$

and the first two elements of the $2k+1$th order of differences obtained from the same series are

$$\sum_{x=0}^{k}(-)^x\left[{2k+1-2x \atop 1}\right]_2 {}^{2k+1}S_{2x} \text{ and } D_k \text{ respectively}$$

Therefore, $$C_{k+1}=\sum_{x=0}^{k}(-)^x\left[{2k+1-2x \atop 1}\right]_2 {}^{2k+1}S_{2x}-a^{2k+1}D_k$$

$$=(a^{2k+1}-1)C_k-a^{2k+1}D_k, \text{ by (2).} \qquad \ldots (5)$$

Again from the series

$$0, -\left[{2k+1 \atop 1}\right]_2, 0, \left[{2k-1 \atop 1}\right]_2, 0, -\left[{2k-3 \atop 1}\right]_2, 0..$$

similarly as (5) we arrive at

$$D_{k+1} = \sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k+1-2x \\ 1 \end{bmatrix}_2 {}^{2k+2}S_{1+2x} + a^{2k+2}\ C_{k+1}$$

$$=(a^{2k+2}-1)D_k + a^{2k+2}\ C_{k+1}\ ,\text{by}(2).$$

Therefore, by (5), we have

$$D_{k+1} = a^{2k+2}(a^{2k+1}-1)C_k - (a^{4k+3}-a^{2k+2}+1)D_k. \qquad \dots \quad (6)$$

Let us now assume that the theorems (3) and (4) both hold good in the kth case, then it can be shown by (5) and (6) that they are also true in the $k+1$th case, but by trial we find that they hold when $k=1$ Therefore they are established by induction.

$$2\quad (i) \qquad {}^{n+1}S_r = a^{r-1}\ {}^nS_{r-1} + a^r\ {}^nS_r. \qquad \dots \quad (7)$$

$$(ii) \qquad {}^{n+1}S_r = {}^nS_r + a^n\ {}^nS_{r-1} \qquad \dots \quad (8)$$

These two theorems may be easily proved by (2).

If $a=1$, each of these theorems reduces to

Cor :— $\qquad {}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r.$

$$(iii) \quad \sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k+1-2x \\ 1 \end{bmatrix}_2 {}^{2k+1}S_{1+2x}$$

$$=(-)^{k-1}(a^{4k+3}-a^{2k+2}-a^{2k+1}+1) \qquad \dots \quad (9)$$

Proof :—

By (7), the left-side

$$= \sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k+1-2x \\ 1 \end{bmatrix}_2 a^{2x}\ {}^{2k}S_{2x}$$

$$+ \sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 (a^{2k+1-2x}-1)a^{1+2x}\ {}^{2k}S_{1+2x}$$

$$=\sum_{x=0}^{k}(-)^{x}\begin{bmatrix}2k+1-2x\\1\end{bmatrix}_{2}a^{2x}\,{}^{2k}S_{2x}$$

$$-\sum_{x=0}^{k-1}(-)^{x}\begin{bmatrix}2k-1-2x\\1\end{bmatrix}_{2}a^{1+2x}\,{}^{2k}S_{1+2x}$$

$$+a^{2k+2}\sum_{x=0}^{k-1}(-)^{x}\begin{bmatrix}2k-1-2x\\1\end{bmatrix}_{2}{}^{2k}S_{1+2x}$$

$$=\sum_{x=0}^{k}(-)^{x}\begin{bmatrix}2k+1-2x\\1\end{bmatrix}_{2}{}^{2k+1}S_{2x}$$

$$+a^{2k+2}\sum_{x=0}^{k-1}(-)^{x}\begin{bmatrix}2k-1-2x\\1\end{bmatrix}_{2}{}^{2k}S_{1+2x},\text{ by (7)}$$

$$=(a^{2k+1}-1)C_{k}+a^{2k+2}(a^{2k}-1)D_{k-1},\text{ by (2).}$$

Hence the theorem is proved.

$$(iv)\quad\sum_{x=0}^{k}(-)^{x}\begin{bmatrix}2k+1-2x\\1\end{bmatrix}_{2}{}^{2k}S_{2x}=(-)^{k-1}(a^{4k}-a^{2k+1}-a^{2k}+1)\qquad(10)$$

Just as (9), the left side is reducible to

$$-(a^{2k}-1)D_{k-1}+a^{2k+1}(a^{2k-1}-1)C_{k-1}.$$

3. $$\sum_{x=0}^{r}(-)^{x}\begin{bmatrix}n-x\\n-x-\delta+1\end{bmatrix}{}^{r}S_{x}$$

$$=0,\ \begin{bmatrix}\delta\\1\end{bmatrix}a^{\delta(n-\delta)}\text{ or }\begin{bmatrix}n-r\\n-\delta+1\end{bmatrix}\begin{bmatrix}\delta\\\delta-r+1\end{bmatrix}a^{r(n-\delta)}\qquad(11)$$

according as δ is $<$, $=$ or $>r$, δ being the number of factors in each term of the summation.

Proof:—

Let us take the series

$$\begin{bmatrix}n\\n-\delta+1\end{bmatrix},\begin{bmatrix}n-1\\n-\delta\end{bmatrix},\begin{bmatrix}n-2\\n-\delta-1\end{bmatrix},\begin{bmatrix}n-3\\n-\delta-2\end{bmatrix},\ \ldots$$

nd obtain from it, by actual calculation, the successive orders of ifferences by using $1, a, a^2, a^3, \ldots$ as multipliers The first elements f the first three orders of differences, thus found, are respectively

$$\begin{bmatrix} n-1 \\ n-\delta+1 \end{bmatrix} (a^{\delta}-1)a^{n-\delta}, \begin{bmatrix} n-2 \\ n-\delta+1 \end{bmatrix}\begin{bmatrix} \delta \\ \delta-1 \end{bmatrix} a^{2(n-\delta)},$$

$$\begin{bmatrix} n-3 \\ n-\delta+1 \end{bmatrix}\begin{bmatrix} \delta \\ \delta-2 \end{bmatrix} a^{3(n-\delta)}$$

Proceeding thus we can deduce the first element of the rth order of differences, *viz.*,

$$\begin{bmatrix} n-r \\ n-\delta+1 \end{bmatrix}\begin{bmatrix} \delta \\ \delta-r+1 \end{bmatrix} a^{r(n-\delta)}$$

which is the right-side of **(11)** when $\delta > r$ but it equals

$$\begin{bmatrix} \delta \\ 1 \end{bmatrix} a^{\delta(n-\delta)} \text{ or zero,}$$

ccording as $\delta =$ or $< r$. But the left-side of (11), is, by (1), the irst element of the rth order of differences obtained from the same eries. So the theorem is proved.

4. $$\sum_{x=0}^{r} (-)^x \frac{1}{a^{n-x}-1}\, {}^{r}S_x = (-)^r \frac{\begin{bmatrix} r \\ 1 \end{bmatrix} a^{\frac{1}{2}r(2n-r-1)}}{\begin{bmatrix} n \\ n-r \end{bmatrix}} \qquad (12)$$

The proof is similar to that of **(11)**, the left-side being the first element of the *rth* order of differences obtained from the series

$$\frac{1}{a^{n}-1}, \frac{1}{a^{n-1}-1}, \frac{1}{a^{n-2}-1}, \frac{1}{a^{n-3}-1}, \ldots$$

by using $1, a, a^2, a^3, \ldots$ as the successive multipliers.

5. Let

$$\phi = \sum_{t=0}^{k} (-)^t\, {}^{k}S_t \begin{bmatrix} r-t \\ r-t-\delta+1 \end{bmatrix} R_t$$

where

$$R_t = \sum_{x=0}^{p} \frac{{}^{t}S_x\, {}^{k-t}S_{p-x}\, a^{(p-x)t}}{a^{r-p-t+x}-1}$$

$r \geq k+p$, $k \geq p$ and $r \geq \delta$, r, k, p and δ being all positive integers.

Then,

(*i*) $\phi=0$, if $\delta<p$;

(*ii*) $\phi=(-)^{k-p}\dfrac{\left[{k \atop 1}\right]a^{kr}}{\left[{r-p \atop r-k}\right]{}^{k+1}S_{k+1}}$, if $\delta=p$;

(*iii*) $\phi=0$, if $\delta>p$ but $<k+1$;

(*iv*) $\phi=\left[{k \atop 1}\right]{}^{k+1}S_p a^{k(r-k-1)}$, if $\delta=k+1$;

(*v*) $\phi={}^kS_p\left[{\delta \atop \delta-p+1}\right]\left[{\delta-p-1 \atop \delta-k}\right]\left[{r-k-1 \atop r-\delta+1}\right]a^{k(r-\delta)}$ if $\delta>k+1$.

The theorem fails if $\delta>r$.

Proof:—

Let us consider the particular case when

$$p=4,\ k=7 \text{ and } r=12$$

Then

$$\phi={}^7S_0\left[{12 \atop 13-\delta}\right]R_0-{}^7S_1\left[{11 \atop 12-\delta}\right]R_1+\ldots-{}^7S_7\left[{9 \atop 6-\delta}\right]R_7,$$

where

$$R_0=\frac{{}^7S_4}{a^8-1},\ R_1=\frac{{}^6S_4a^4}{a^7-1}+\frac{{}^6S_3a^8}{a^8-1},$$

$$R_2=\frac{{}^5S_4a^8}{a^6-1}+\frac{{}^9S_1\,{}^5S_3a^6}{a^7-1}+\frac{{}^9S_2\,{}^5S_2a^4}{a^8-1},$$

$$R_3=\frac{{}^4S_4a^{12}}{a^5-1}+\frac{{}^6S_1\,{}^4S_3a^9}{a^6-1}+\frac{{}^6S_2\,{}^4S_2a^6}{a^7-1}+\frac{{}^6S_3\,{}^4S_1a^3}{a^8-1},\ldots R_7=\frac{{}^7S_4}{a^5-1},$$

Hence in ϕ, the coefficient of $\dfrac{1}{a^{8-q}-1}$ (q varies from 0 to 3)

$$=\sum_{x=0}^{4}(-)^{q+x}\,{}^7S_{q+x}\left[{12-q-x \atop 13-q-x-\delta}\right]{}^{7-q-x}S_{4-x}\,{}^{q+x}S_x a^{(q+x)(4-x)}$$

$$=(-)^q\,{}^7S_{4+q}\,{}^{4+q}S_q\,a^{-\frac{1}{2}q(q-1)}\sum_{x=0}^{4}(-)^x\left[{12-q-x \atop 13-q-x-\delta}\right]{}^4S_x, \text{ by (2)}, \quad (13)$$

But by (11),

$$\sum_{x=0}^{4}(-)^x\begin{bmatrix}12-q-x\\13-q-x-\delta\end{bmatrix}{}^4S_x=0,\ \begin{bmatrix}\delta\\1\end{bmatrix}a^{\delta(12-q-\delta)}$$

$$\text{or}\begin{bmatrix}8-q\\13-q-\delta\end{bmatrix}\begin{bmatrix}\delta\\\delta-3\end{bmatrix}a^{4(12-q-\delta)} \quad \ldots \quad (14)$$

according as δ is $<,=$ or >4.

(*i*) From (13), we get the coefficients of $\frac{1}{a^8-1}, \frac{1}{a^7-1}, \frac{1}{a^6-1}$, and $\frac{1}{a^5-1}$ each of which vanishes by (14) when $\delta<4$.

Therefore, $\phi=0$, if $\delta=1, 2$ or 3

(*ii*) If $\delta=4$, from (13) and (14) we have

$$\phi=\sum_{q=0}^{3}(-)^q\ {}^7S_{4+q}\,{}^{4+q}S_q a^{-\frac{1}{2}q(q-1)}\begin{bmatrix}4\\1\end{bmatrix}a^{4(8-q)}\frac{1}{a^{8-q}-1}$$

$$={}^7S_4\begin{bmatrix}4\\1\end{bmatrix}a^{32}\sum_{q=0}^{3}(-)^q\frac{1}{a^{8-q}-1}\ {}^3S_q,\text{ by (2)}$$

But by (12),

$$\sum_{q=0}^{3}(-)^q\frac{1}{a^{8-q}-1}{}^3S_q=(-)^3\frac{\begin{bmatrix}3\\1\end{bmatrix}a^{18}}{\begin{bmatrix}8\\5\end{bmatrix}}$$

$$\therefore\phi=-\frac{\begin{bmatrix}4\\1\end{bmatrix}a^{50}}{a^5-1},\text{ if }\delta=4.$$

(*iii, iv, v*) If $\delta>4$, by (13) and (14), we have

$$\phi=\sum_{q=0}^{3}(-)^q\ {}^7S_{4+q}\ a^{-\frac{1}{2}q(q-1)}\begin{bmatrix}8-q\\13-q-\delta\end{bmatrix}$$

$$\times\begin{bmatrix}\delta\\\delta-3\end{bmatrix}a^{4(12-q-\delta)}\frac{1}{a^{8-q}-1}$$

$$=\begin{bmatrix}\delta\\\delta-3\end{bmatrix}{}^7S_4\ a^{4(12-\delta)}\sum_{q=0}^{3}(-)^q\begin{bmatrix}7-q\\13-q-\delta\end{bmatrix}{}^3S_q,\text{ by (2)} \quad \ldots \quad (15)$$

But by (11),

$$\sum_{q=0}^{3} (-)^q \begin{bmatrix} 7-q \\ 13-q-\delta \end{bmatrix} {}^3S_q = 0, \begin{bmatrix} 3 \\ 1 \end{bmatrix} a^{12} \text{ or} \begin{bmatrix} 4 \\ 13-\delta \end{bmatrix} \begin{bmatrix} \delta-5 \\ \delta-7 \end{bmatrix} a^{3(12-\delta)} \tag{16}$$

according as the number of factors in each term of the summation, *viz.*,

$\delta-5 <, = \text{or} > 3$ *i.e.*, as $\delta <, = \text{or} > 8$.

Hence

$$\phi = 0, \text{ if } \delta = 5, 6 \text{ or } 7$$

The values of ϕ when $\delta = \text{or} > 8$ readily follow from (15) and (16).

In the general case we are first to write out R_0, R_1, R_2, R_3, R_4, ... R_p, $R_{p+1}R_{p+2}$ and then from ϕ, pick up separately the coefficients of

$$\frac{1}{a^{r-p}-1}, \frac{1}{a^{r-p-1}-1}, \frac{1}{a^{r-p-2}-1}, \frac{1}{a^{r-p-3}-1}, \ldots$$

These coefficients will enable us to deduce the coefficient of $\dfrac{1}{a^{r-p-q}-1}$ (q varies from 0 to $k-p$), which is

$$= \sum_{x=0}^{p} (-)^{q+x} \begin{bmatrix} r-q-x \\ r-q-x-\delta+1 \end{bmatrix} {}^kS_{q+x} {}^{q+x}S_x$$

$$\times {}^{k-q-x}S_{p-x}\, a^{(q+x)(p-x)}$$

$$= (-)^q\, {}^kS_{p+q} {}^{p+q}S_q\, a^{-\frac{1}{2}q(q-1)} \sum_{x=0}^{p} (-)^x \begin{bmatrix} r-q-x \\ r-q-x-\delta+1 \end{bmatrix} {}^pS_x, \text{ by (2).}$$

Then proceed just as in the particular case given above.

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On a Problem in the Stability of a Circular Vortex

BY

MANOHAR RAY

(*Calcutta*)

1. *Introduction* :—Lord Kelvin* has examined the question of stability of a circular vortex surrounded by an infinite fluid moving irrotationally, in the case when the vorticity is uniform and has shown that the motion is stable when the disturbance consists of a system of corrugations travelling round the circumference of the vortex. In the present note firstly, it is pointed out that the stability is unaffected for such disturbance even when the vorticity is a function of the distance from the centre of the vortex. Secondly, a modified problem is attempted when the vortex surrounds a concentric cylindrical obstacle which is forced to execute small vibration. The question is whether such a configuration is possible or the system will break up. It is shown that if the inner cylinder executes small circular vibration the system may be stable only when the vorticity is uniform. It has also been possible to calculate the forces necessary to maintain the vibration. The expressions resemble Blasius's force components on a fixed cylinder in a uniform stream with circulation.

2. *Stability of a circular vortex with non-uniform vorticity* :—

In the first place, let us suppose that the space inside the circle $r=a$, having the centre as origin, is occupied by fluid having a vorticity which depends on the distance from the centre, and that this circular vortex is surrounded by fluid moving irrotationally. Then we require the solution of the equation

$$\nabla^2\psi=2\zeta, \qquad \dots \quad (1)$$

where ζ is a function of r only.

* Sir W. Thomson, "On the Vibration of a Columnar Vortex," Phil. Mag. (5) X, 155 (1880).

In polar coordinates, (1) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)=2\zeta,$$

i.e.,
$$r\frac{d\psi}{dr}=2\int_0^r \zeta r dr=rf(r), \text{ say} \qquad \dots \quad (2)$$

so that
$$\psi=-[F(a)-F(r)],$$

where
$$\int^r f(r)dr=F(r). \qquad \dots \quad (3)$$

Now from (2)

$$2\zeta r=\frac{d}{dr}\Big(r\,f\,(r)\Big)=f(r)+rf'(r),$$

and if ω_r be the angular velocity of rotation at distance r

$$r\omega_r=f(r),$$

so that
$$2\zeta-\omega_r=f'(r). \qquad \dots \quad (4)$$

Hence we assume,

for $r<a$,

$$\psi=-[F(a)-F(r)] \qquad \dots \quad (5)$$

and for $r>a$,

$$\psi=-B\log\frac{a}{r}. \qquad \dots \quad (6)$$

The assumptions (5) and (6) give the radial component of velocity, $-\frac{\partial\psi}{r\partial\theta}$, zero on both sides of the circle $r=a$, and in order that the transverse component $\frac{\partial\psi}{\partial r}$ may be continuous on $r=a$, we must have from (5) (6) and (3), on $r=a$,

$$\frac{B}{r}=F'(r)=f(r),$$

$$\text{i.e., } B=af(a). \qquad \dots \quad (7)$$

Thus the constant B in (6) is determined.

Giving the system a slight irrotational disturbance, we take,

for $r<a$, $$\psi=-[F(a)-F(r)]+C\frac{r^s}{a^s}\cos(s\theta-\sigma t), \quad \dots \quad (8)$$

and for $r>a$, $$\psi=-B\log\frac{a}{r}+C\frac{a^s}{r^s}\cos(s\theta-\sigma t), \quad \dots \quad (9)$$

where s is integral and C very small.

The above assumptions (8) and (9) evidently make the radial component of velocity, $-\dfrac{\partial\psi}{r\,\partial\theta}$, continuous at the boundary of the vortex, for which $r=a$ approximately. To examine the continuity of the transverse component of velocity, $\dfrac{\partial\psi}{\partial r}$, we take for the equation of the boundary

$$r=a+\alpha\cos(s\theta-\sigma t),$$

where α is very small,

i.e., $$r=a+\xi, \quad \dots \quad (10)$$

where ξ is small.

Thus we must have on (10),

$$f(r)+Cs\frac{r^{s-1}}{a^s}\cos(s\theta-\sigma t)=\frac{B}{r}-Cs\frac{a^s}{r^{s+1}}\cos(s\theta-\sigma t),$$

i.e., $$f(a+\xi)+\frac{Cs}{a}\cos(s\theta-\sigma t)=\frac{B}{a}\left(1-\frac{\xi}{a}\right)-\frac{Cs}{a}\cos(s\theta-\sigma t)$$

or $$f(a)+\xi f'(a)+\dots+\frac{Cs}{a}\cos(s\theta-\sigma t)=\frac{B}{a}-\frac{B\xi}{a^2}-\frac{Cs}{a}\cos(s\theta-\sigma t)$$

or, using (7) and substituting for ξ,

$$2\frac{Cs}{a}=-\frac{B\alpha}{a^2}-\alpha f'(a)=-\left\{\frac{B}{a^2}+f'(a)\right\}\alpha. \quad \dots \quad (11)$$

We are still left with the dynamical condition that the vortex-lines move with the fluid which requires that the normal velocity of a particle on the boundary must be equal to that of the boundary itself. This condition gives

$$\frac{\partial r}{\partial t}=-\frac{1}{r}\frac{\partial\psi}{\partial\theta}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial\theta}, \quad \dots \quad (12)$$

where r has the value (10)

Thus from (8) and (10), we get

$$a\sigma \sin(s\theta-\sigma t)=Cs\,\frac{r^{s-1}}{a^s}\sin(s\theta-\sigma t)$$

$$+\left[F'(r)+Cs\frac{r^{s-1}}{a^s}\cos(s\theta-\sigma t)\right]\frac{as}{r}\sin(s\theta-\sigma t),$$

whence neglecting quadratic terms in a and C,

$$a\sigma=\frac{Cs}{a}+F'(a)\frac{as}{a}=\frac{Cs}{a}+\frac{f(a)}{a}as, \quad \dots \quad (13)$$

from (3)

Thus from (7), (11) and (13), we get, substituting for B and C,

$$\sigma=-\tfrac{1}{2}\left\{\frac{f(a)}{a}+f'(a)\right\}+\frac{f(a)}{a}s$$

$$\text{i.e.,}\ \sigma=-\zeta_a+\frac{f(a)}{a}s, \quad \dots \quad (14)$$

where ζ_a is the value of ζ on $r=a$

If ω_a be the angular velocity on the rim $r=a$,

$$\omega_a=\frac{f(a)}{a},$$

hence $$\sigma=s\omega_a-\zeta_a \quad \dots \quad (15)$$

The angular velocity of the sinuous waves of disturbance is

$$\frac{\sigma}{s}=\omega_a-\frac{\zeta_a}{s} \quad \dots \quad (16)$$

Hence the circular vortex with a general law of vorticity symmetrical with respect to r is stable for a circularly travelling disturbance of irrotational type.

If the vorticity be uniform, *i.e.*, $2\zeta=\text{constant}=\omega$, then (2) gives,

$$f(r)=\zeta r=\frac{\omega}{2}\,r, \text{ so that } \omega_a=\frac{\omega}{2}$$

hence (16) gives

$$\frac{\sigma}{s}=\frac{\omega}{2}\left(1-\frac{1}{s}\right). \qquad \dots \quad (17)$$

This result is the same as that found by Lord Kelvin.*

3 *Vibration of a circular cylinder inside a circular vortex* :—

Next let us suppose that a cylindrical body with cross-section $r=b$ $(b<a)$ vibrates with velocity $(U\cos nt,\ U\sin nt)$ where U is small, inside the circular vortex.

If u_r, u_θ be the components of velocity radially and transversely at any point, due to the presence of the vortex only, p the pressure, the equations of motion are

$$\left.\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial r} &= \tfrac{1}{2}\frac{\partial}{\partial r}\,(u_r^2+u_\theta^2)-2\zeta u_\theta, \\ -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} &= \frac{1}{2r}\frac{\partial}{\partial \theta}\,(u_r^2+u_\theta^2)+2\zeta u_r, \end{aligned}\right\} \qquad \dots \quad (18)$$

where
$$2\zeta=\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}, \qquad \dots \quad (19)$$

and
$$u_r=-\frac{1}{r}\frac{\partial \psi}{\partial \theta},\ u_\theta=\frac{\partial \psi}{\partial r}, \qquad \dots \quad (20)$$

which satisfy the equation of continuity.

If u'_r, u'_θ and p' be the contributions due to the vibration of the inner cylinder, so that u'_r, u'_θ and p' are all small, they satisfy the equations,

* H. Lamb, Hydrodynamics, 6th Edition, 1932, p 231.

$$\left.\begin{aligned}-\frac{1}{\rho}\frac{\partial}{\partial r}(p+p')&=\frac{\partial u'_r}{\partial t}+\tfrac{1}{2}\frac{\partial}{\partial r}(u_r^2+u_\theta^2+2u_ru'_r+2u_\theta u'_\theta)-2\zeta(u_\theta+u'_\theta),\\ -\frac{1}{\rho}\frac{\partial}{r\partial\theta}(p+p')&=\frac{\partial u_\theta'}{\partial t}+\frac{1}{2r}\frac{\partial}{\partial\theta}(u_r^2+u_\theta^2+2u_ru'_r+2u_\theta u'_\theta)+2\zeta(u_r+u'_r),\end{aligned}\right\}.\quad(21)$$

neglecting the quadratic terms in u'_r, u'_θ and assuming that the vibration of the inner cylinder produces only an irrotational disturbance.

Thus from (21) and (18), we get

$$-\frac{1}{\rho}\frac{\partial p'}{\partial r}=\frac{\partial u'_r}{\partial t}+\frac{\partial}{\partial r}(u_ru'_r+u_\theta u'_\theta)-2\zeta u'_\theta \quad\dots\quad(22)$$

$$-\frac{1}{\rho}\frac{1}{r}\frac{\partial p'}{\partial\theta}=\frac{\partial u_\theta'}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(u_ru'_r+u_\theta u'_\theta)+2\zeta u'_r \quad\dots\quad(23)$$

where $u'_r=-\dfrac{1}{r}\dfrac{\partial\psi'}{\partial\theta}$, $u'_\theta=\dfrac{\partial\psi'}{\partial r}$, ψ' being the addition to ψ due to the vibration of the inner cylinder.

But $u_r=0$ and u_θ is a function of r only, as seen form (5) and (6) so that we get the following equations,

$$\left.\begin{aligned}-\frac{1}{\rho}\frac{\partial p'}{\partial r}&=\frac{\partial u'_r}{\partial t}+\frac{\partial}{\partial r}(u_\theta u'_\theta)-2\zeta u'_\theta\\ -\frac{1}{\rho}\frac{1}{r}\frac{\partial p'}{\partial\theta}&=\frac{\partial u'_\theta}{\partial t}+\frac{u_\theta}{r}\frac{\partial u'_\theta}{\partial\theta}+2\zeta u'_r.\end{aligned}\right\}\quad\dots\quad(24)$$

Let us now assume,

$$u'_r=u''_r\cos(\theta-nt),\ u'_\theta=u''_\theta\sin(\theta-nt),\ p'=p''\sin(\theta-nt)$$

where u''_r, u''_θ, p'' are functions of r only.

Then the equations (24) give

$$\left.\begin{aligned}-\frac{1}{\rho}\frac{\partial p''}{\partial r}&=nu''_r+\frac{d}{dr}(u_\theta u''_\theta)-2\zeta u''_\theta\\ -\frac{1}{\rho}\frac{p''}{r}&=-nu''_\theta+\frac{u_\theta u_\theta''}{r}+2\zeta u''_r.\end{aligned}\right\}\quad\dots\quad(25)$$

Now ψ' satisfies the equation

$$\frac{\partial^2\psi'}{\partial r^2}+\frac{1}{r}\frac{\partial\psi'}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi'}{\partial\theta^2}=0, \qquad \dots \quad (26)$$

so that putting $\psi'=\psi''\sin(\theta-nt)$ where ψ'' is a function of r only, the equation for ψ'' is

$$\frac{d^2\psi''}{dr^2}+\frac{1}{r}\frac{d\psi''}{dr}-\frac{\psi''}{r^2}=0, \qquad \dots \quad (27)$$

and

$$u''_r=-\frac{\psi''}{r},\ u''_\theta=\frac{d\psi''}{dr}. \qquad \dots \quad (28)$$

Substituting these values of u''_r and u''_θ in (25) we have

$$-\frac{1}{\rho}\frac{dp''}{dr}=-\frac{n\psi''}{r}+\frac{d}{dr}(u_0u''_\theta)-2\zeta\frac{d\psi''}{dr} \qquad \dots \quad (25a)$$

$$-\frac{p''}{\rho}=-nr\frac{d\psi''}{dr}+u_\theta u''_\theta-2\zeta\psi''. \qquad \dots \quad (25b)$$

Differentiating (25b) with respect to r and using (27) we find that the resulting equation is consistent with (25a) only when

$$\frac{d\zeta}{dr}=0,$$

i.e., $\zeta=$ constant. ... (29)

Thus the condition of consistency of (25) and (27) requires that the vorticity should be uniform.

Then from (25b),

$$\frac{p'}{\rho}=nr\frac{\partial\psi'}{\partial r}-u_\theta u'_\theta+2\zeta\psi'=(nr-u_\theta)\frac{\partial\psi'}{\partial r}+2\zeta\psi'. \qquad \dots \quad (30)$$

Again, since u_r is zero and u_θ is a function of r only, equations (18) give

$$\frac{p}{\rho}=-\frac{u_\theta^2}{2}+2\zeta\psi. \qquad \dots \quad (31)$$

Thus the complete value of the pressure at any point is obtained by adding together (30) and (31) and is given by

$$\frac{p}{\rho}=\frac{p_0}{\rho}-\frac{u_\theta^2}{2}+2\zeta\psi+(nr-u_\theta)\frac{\partial\psi'}{\partial r}+2\zeta\psi', \quad \text{..} \quad (32)$$

where p_0 is some constant and $u_\theta=\frac{d\psi}{dr}$.

4. *Determination of ψ and ψ'*:—

Now ψ is the stream function due to the presence of the vortex only and ψ' due to the vibration of the inner cylinder, so that the complete stream-function is given by

$$\Psi=\psi+\psi'. \quad \text{...} \quad (33)$$

Since ζ is constant, let us assume,*

for $b<r<a$, $$\psi=-\tfrac{1}{2}\zeta(a^2-r^2)+(Ar+\frac{B}{r})\sin(\theta-nt), \quad \text{...} \quad (34)$$

and for $r>a$, $$\psi=-\zeta a^2\log\frac{a}{r}+\frac{C}{r}\sin(\theta-nt), \quad \text{...} \quad (35)$$

where A, B and C are small constants

The boundary conditions to be satisfied are,

on $r=b$, $-\frac{1}{r}\frac{\partial\psi}{\partial\theta}=U\cos nt\cos\theta+U\sin nt\sin\theta=U\cos(\theta-nt)$, (36)

on $r=a$, $-\frac{1}{r}\frac{\partial\psi}{\partial\theta}$ is to be continuous, ... (37)

on $r=a+\alpha\sin(\theta-nt)$, where α is small,

$\frac{\partial\psi}{\partial r}$ is to be continuous, ... (38)

$$\text{and } \frac{\partial r}{\partial t}=-\frac{1}{r}\frac{\partial\psi}{\partial\theta}-\frac{\partial\Psi}{\partial r}\frac{1}{r}\frac{\partial r}{\partial\theta}; \quad \text{...} \quad (39)$$

at infinity the velocity is to be zero, which is evident from (35).

* Lamb, Hydrodynamics, already mentioned

The conditions (36) and (37) give

$$Ab^2+B=-Ub^2, \quad \ldots \quad (40)$$

$$Aa^2+B=C \quad \ldots \quad (41)$$

From (38), using (41), we get

$$2\zeta a=-2A, \quad \ldots \quad (42)$$

and the condition (39), with the help of (42), gives

$$an=\frac{B}{a^2}. \quad \ldots \quad (43)$$

From (40), (41), (42) and (43), we get the following values for the constants A, B, C, namely,

$$\left.\begin{aligned} A&=-\frac{Ub^2\zeta}{\zeta b^2-na^2}, \\ B&=\frac{Ua^2b^2n}{\zeta b^2-na^2}, \\ C&=\frac{(n-\zeta)Ua^2b^2}{\zeta b^2-na^2}. \end{aligned}\right\} \quad \ldots \quad (44)$$

Substituting these values of A, B, C we get the stream-function Ψ from the equations (34) and (35). This shows that the original circular vortex will not break up but vibrate in unison with the central cylinder (which is supposed to produce only irrotational disturbance) provided the vorticity is uniform.

5. *Calculation of the Resistance* :—

From the solutions obtained above we can calculate the resistance or the force necessary (up to the first order) to maintain the vibration of the cylinder.

For $b<r<a$, $\psi=-\frac{1}{2}\zeta(a^2-r^2)$, $\psi'=\left(Ar+\frac{B}{r}\right)\sin(\theta-nt)$,

so that $u_\theta=\frac{d\psi}{dr}=\zeta r$ and $\frac{\partial\psi'}{\partial r}=\left(A-\frac{B}{r^2}\right)\sin(\theta-nt)$,

hence from (32),

$$\frac{p}{\rho}=\frac{p_0}{\rho}-\frac{\zeta^2 r^2}{2}-\zeta^2(a^2-r^2)+(nr-\zeta r)\left(A-\frac{B}{r^2}\right)\sin(\theta-nt)$$

$$+2\zeta\left(Ar+\frac{B}{r}\right)\sin(\theta-nt) \quad \ldots \quad (45)$$

If F_1 and F_2 be the components of the resistance experienced by the inner cylindrical body $r=b$, due to its vibration, then

$$F_1=-\int_0^{2\pi}(p)_{r=b}\cos\theta\, bd\theta \text{ and } F_2=-\int_0^{2\pi}(p)_{r=b}\sin\theta\, bd\theta. \quad \ldots \quad (46)$$

Substituting for p from (45) and putting in the values of the constants A, B, C, we get at once,

$$\left.\begin{aligned} F_1&=-\pi\rho\left[\frac{(n-\zeta)(b^2\zeta+na^2)}{b^2\zeta-na^2}+2\zeta\right]b^2U\sin nt,\\ F_2&=+\pi\rho\left[\frac{(n-\zeta)(b^2\zeta+na^2)}{b^2\zeta-na^2}+2\zeta\right]b^2U\cos nt.\end{aligned}\right\} \quad \ldots \quad (47)$$

That is,

$$\left.\begin{aligned} F_1&=K_\rho V_2,\\ F_1&=-K_\rho V_1,\end{aligned}\right\} \quad \ldots \quad (48)$$

where (V_1, V_2) is the velocity of the inner cylinder $r=b$

and $$K=-\pi b^2\left[\frac{(n-\zeta)(\zeta b^2+na^2)}{\zeta b^2-na^2}+2\zeta\right].$$

This result resembles the Blasius's expressions for the force-components on a fixed cylinder in a uniform circulatory stream. In particular when the angular velocity of the waves of disturbance relative to the rotating fluid is zero, *i.e.*, $n=\zeta$, $K=-2\pi b^2\zeta$, which can be regarded as the strength of the portion of the vortex displaced by the cylinder.

In conclusion, I want to express my gratefulness to Prof. N. R. Sen for his kind help in this work.

6

A Note on the Convex Oval

BY

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Introduction.

Corresponding to special properties of the ellipse it is often possible to find properties of the closed convex oval. The object of the present paper is to investigate the properties of the oval corresponding to the following properties of the ellipse:—(1) At a point O interior to the ellipse just one chord is bisected, unless O is the centre of the ellipse when an infinite number of chords is bisected. (2) If O is a point interior to the ellipse, there exists just one pair of parallel tangents equidistant from O, unless O is the centre of the ellipse when an infinite number of such pairs exist.

Corresponding to the property (1) I prove:—

Theorem (A). *If O is any point within a closed convex oval, then if a finite number of chords is bisected at O, this number must be odd.*

Theorem (B). *At least three distinct chords are bisected at the centre of mass of the area of the oval.*

The property (2) of the ellipse may be looked upon as the dual of the property (1). In fact I prove;

Theorem (C). *The number of chords bisected at a point O within a closed convex oval is exactly equal to the number of pairs of parallel tangents equidistant from O.*

From this it follows at once with the help of the previous theorems:

Theorem (D). *If O is any point within a closed convex oval and if there exists a finite number of pairs of parallel tangents equidistant from O, this number must be odd.*

Theorem (E). *On an oval V there are at least three pairs of points, such that the tangents at each pair are parallel, and the distances of the tangents from the centre of mass of the area of the oval are equal.*

Steiner* has defined the curvature centroid of an oval, as the centre of mass of the perimeter of the oval, if every point of the perimeter is considered to have a density equal to the curvature at the point. Hayashi has proved that the property of the Theorem (E) holds also for the curvature centroid †

From this it follows at once with the help of Theorem (C) ;—

Theorem (F). *At least three distinct chords of a closed convex oval V are bisected at the curvature centroid of V.*

I.

1. Let V be closed convex oval, and O a point to the interior of the oval In counting the number of chords of V bisected at a point O we shall adopt the following convention ;—Consider a chord POQ of V passing through O and turning about O, P and Q describing the oval. If as POQ passes through the particular position, P_0OQ_0, the algebraic difference OP—OQ vanishes and changes sign, P_0OQ_0 counts as a single chord bisected at O, while if the algebraic difference OP—OQ vanishes but does not change sign, P_0OQ_0 counts as two chords bisected at O. This convention may be analytically expressed in the following way :—If $r=f(\theta)$ is the equation of V with reference to O as the pole, and a suitably chosen line as the initial line, and if θ and $\theta+\pi$ are the vectorial angles of P and Q then if the function $f(\theta)-f(\theta+\pi)$ vanishes and changes sign at $\theta=\theta_0$, the chord joining the points whose vectorial angles are θ_0 and $\theta_0+\pi$ counts as a single chord bisected at O, while if the same function vanishes but does not change sign at $\theta=\theta_0$, the chord under consideration counts as two chords bisected at O

2. We shall first prove the following Lemma.—

Lemma *If* $F(\theta)$ *is a continuous periodic function of* θ *with period* 2π *having the property* $F(\theta)+F(\theta+\pi)=0$, *then if* $F(\theta)=0$ *has only a finite number of roots in a complete period, it must vanish and change sign at an odd number of points in the half period* $(0, \pi-0)$.

* J Steiner. Von dem Krummungsschwerpunkte ebener Kurven Crelle, J 21 (1888)

† T. Hayashi Rend, Circ. Matem, T, L (1926).

In the first place suppose $F(0)$ is $+ve$ Then $F(\pi)$ is $-ve$.

Let $\theta_1, \theta_2, \ldots \ldots \theta_n$ be all the points in $(0, \pi-0)$ at which $F(\theta)$ vanishes and changes sign We have to prove that n is odd. In each of the $(n+1)$ intervals $(0, \theta_1)$, $(\theta_1, \theta_2) \ldots \ldots$, (θ_{n-1}, θ_n), $(\theta_n, \pi-0)$, $F(\theta)$ maintains sign except that it may vanish at a finite number of points within any interval The sign of $F(\theta)$ in the first interval $(0, \theta_1)$ is $+ve$, being the same as the sign of $F(0)$, while the sign of $F(\theta)$ in the last interval $(\theta_n, \pi-0)$ is $-ve$ being the same as the sign of $F(\pi)$ Now the signs of $F(\theta)$ in the $(n+1)$ intervals under consideration are alternately positive and negative, thus the sign in the last interval cannot be negative unless $(n+1)$ is even, or n is odd

In case $F(0)$ is $-ve$, $F(\pi)$ will be positive and a similar proof applies.

When $F(0)=0$, then since the number of roots of $F(\theta)=0$, has been supposed to be finite we can find a value α such that $F(\alpha)\neq 0$. Let $F_1(\theta)=F(\theta+\alpha)$ Then from what has been proved $F_1(\theta)=0$, has an odd number of roots in $(0, \pi-0)$, so that $F(\theta)=0$ has an odd number of roots in $(\alpha, \pi+\alpha-0)$. But from the property $F(\theta)+F(\theta+\pi)=0$ it follows that corresponding to any root $\pi+\theta_1$ of $F(\theta)=0$ in the interval $(\alpha, \pi+\alpha-0)$ there exists a root θ_1 in the interval $(0, \alpha-0)$ Consequently $F(\theta)=0$ will also have an odd number of roots in $(0, \pi-0)$.

We have thus completely proved our Lemma.

3. **Theorem (A).** *If O is any point within a closed convex oval V, then if a finite number of chords is bisected at O, this number must be odd.*

Let $r=f(\theta)$ be the equation of V with the respect to O as the pole Then if a chord $P_0\,O\,Q_0$ is bisected at O, one and only one of its extremities (say P_0) will have its vectorial angle θ_0 lying in the interval $(0, \pi-0)$. The chord, however, counts as a single chord or a double chord according as $F(\theta)\equiv f(\theta)-f(\theta+\pi)$ vanishes at θ_0 changing sign at the same time or vanishes at θ_0 without changing sign. The number of single chords of V bisected at O is thus exactly equal to the number of roots of $F(\theta)=0$, in the interval $(0, \pi-0)$ at which $F(\theta)$ vanishes and changes sign. Now $F(\theta)$ obviously satisfies the conditions of our Lemma. Consequently the number of single chords bisected at O is odd. The total number of chords bisected at O, must therefore also be odd, since other chords bisected at O, count as two

chords, and therefore do not matter, so far as evenness or oddness is concerned.

Corollary. *If two chords are bisected at a point O within a closed convex oval, there must exist a third chord which is bisected at the point.*

4 **Theorem (B).** *At least three distinct chords are bisected at the centre of mass of the area of an oval*

Let O be the centre of mass of the area of the oval It follows from Theorem (A), that there must exist at least a single chord P_0OQ_0 of V, which is bisected at O and counts as a single chord bisected at O. Let $r=f(\theta)$ be now the equation of V, referred to O as the pole and OP_0 as the initial line. Setting as before $F(\theta)=f(\theta)-f(\theta+\pi)$ it follows that $F(0)=0$ we shall first show that $F(\theta)$ must vanish and change sign, at least at one interior point of the interval $(0, \pi)$

Now the distance of the C. G. of V from the initial line is given by

$$\frac{1}{3A}\int_0^{2\pi}\{f(\theta)\}^3 \sin\theta d\theta,$$

where A is the area of the oval. Since the C G lies on the initial line itself

$$\int_0^{2\pi}\{f(\theta)\}^3 \sin\theta d\theta=0,$$

or by an easy transformation

$$\int_0^{\pi}[\{f(\theta)\}^3-\{f(\theta+\pi)\}^3] \sin\theta d\theta=0 \qquad \dots \quad \text{(i)}$$

Now if $F(\theta)=f(\theta)-f(\theta+\pi)$ does not vanish and change sign at an interior point of the interval $(0, \pi)$, the integrand in (i) maintains an invariable sign in the interior of the interval of integration. This, however, makes the equation (i) impossible. Consequently there exists a number θ_1, $0<\theta_1<\pi$ such that $F(\theta)$ vanishes and changes sign at $\theta=\theta_1$. From our Lemma then follows the existence of a third number θ_2, $0<\theta_2<\pi$, $\theta_1\neq\theta_2$, at which $F(\theta)$ vanishes and changes sign. Let P_1, P_2 be points on V with vectorial angles θ_1 and θ_2 and Q_1, Q_2 the points of V with vectorial angles $\pi+\theta_1$, $\pi+\theta_2$ Then the three distinct chords P_0OQ_0, P_1OQ_1, P_2OQ_2 are bisected at O.

II.

1. We shall now consider the number of pairs of parallel tangents whose members are equidistant from a point O interior to the convex oval V. In counting the number of such pairs we shall adopt a convention similar to that adopted in counting the number of chords bisected at O. Let t be any tangent to V and τ the parallel tangent. Let L and M be the feet of the perpendiculars from O upon t and τ. Let now t turn remaining tangential to V. If as t passes through a particular position t_0, the algebraic difference OL—OM vanishes and changes sign, $t_0\tau_0$ counts as one pair of parallel tangents equidistant from O, while if OL—OM vanishes but does not change sign then $t_0\tau_0$ counts as two pairs of parallel tangents equidistant from O.

2. We shall now prove the following theorem :—

Theorem (C) *The number of chords bisected at a point O within a closed convex oval V, is equal to the number of pairs of parallel tangents equidistant from O.*

The reflection of any point or line in the plane, in the point O, we shall denote by placing a dash on the letter denoting the point or the line. Thus P′ denotes the reflection of the point P in O, while t' denotes the reflection of the line t in O. Now as P describes the oval V, P′ describes another oval V′ which is the reflection of V in O. If t is the tangent to V at P, t' is the tangent at P′ to V′. Now if POQ is a chord of V bisected at O, then P′ coincides with Q and Q′ coincides with P, so that V′ meets V at the points P and Q. When POQ counts as a single chord bisected at O, according to our convention it is easy to see that V′ crosses V at P and Q, while if POQ counts as two chords bisected at O, according to our convention then V′ touches V from within at one of the points P and Q, while it touches V from without at the other point. Thus if a common point of V and V′ counts as a single intersection or a double intersection of V and V′, according as V and V′ cross at the point, or touch without crossing we can assert that the number of intersections of V and V′ is exactly double the number of chords of V bisected at O. In the same way if a common tangent of V and V′ counts as one or two according as its point of contact with V is not, or is coincident, with its point of contact with V′, we can assert that the number of common tangents of V and V′ is exactly double the number of pairs of parallel tangents of V equidistant from O. But as V and V′ are closed convex ovals, the number

of their intersections is exactly equal to the number of their common tangents. Hence our theorem follows

3. From Theorems (A) and (B) we now derive —

Theorem (D) *If O is any point within a closed convex oval V and if there exists a finite number of pairs of parallel tangents equidistant from O, then this number must be odd*

Theorem (E) *On an oval V there are at least three pairs of points, such that the tangents at each pair are parallel and the distances of the tangents from the centre of mass of the area of the oval are equal.*

Hayashi has shown that the property of the Theorem (E) is true also with respect to the curvature centroid of the oval.* We deduce at once

Theorem (F) *At least three distinct chords of a closed convex oval V are bisected at the curvature centroid of* V.

In conclusion my thanks are due to Professor Dr S Mukhopadhyaya who suggested the investigation to me

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* T Hayashi, *Loc cit*

7

Flexure of Beams of certain Forms of Cross-sections

BY

S. Ghosh.

Although the torsion problem has been worked out for a large number of cases, the solution of the flexure problem is known for a comparatively few boundaries only. As far as I am aware, the only two known cases, where elliptic boundaries appear, are (1) when the cross-section consists of an ellipse, and (2) when it consists of two confocal ellipses. In the present paper, I have given the solution of the flexure problem for a beam whose cross-section consists of (1) a semi-ellipse, bounded by its minor axis, and (2) an ellipse and two equal confocal hyperbolas. Finally, the second of these cases has been reduced to the interesting case of an elliptic beam, with two cracks extending from the foci to the boundary of the ellipse, along its major axis.

We take the origin at the centroid of the fixed end of the beam, and the line of centroids of the cross sections as the axis of z, which we consider to be horizontal. The axis of x is taken vertically downwards and the axis of y horizontal. Further, the axes of x and y are assumed to be parallel to the principal axes of inertia at the centroids of the cross-sections. The load W acts vertically downwards and is applied at the centroid of the other end of the beam.

Omitting rigid body displacements, the displacements are given by*

$$\left.\begin{aligned}
u &= -\tau yz + \frac{W}{EI}\left[\tfrac{1}{2}(l-z)\sigma(x^2-y^2)+\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right],\\
v &= \tau zx + \frac{W}{EI}\sigma(l-z)\,xy,\\
w &= \tau\phi - \frac{W}{EI}\left[x(lz-\tfrac{1}{2}z^2)+\chi+xy^2\right],
\end{aligned}\right\} \quad \cdots \quad (1)$$

* Love, 'The Mathematical Theory of Elasticity' (4th ed.), p. 334.

where I is the moment of inertia of a cross section about its principal axis parallel to oy and E and σ denote Young's modulus and Poisson's ratio for the material of the beam.

ϕ is the torsion function for the section and χ is a function independent of z, which satisfies the equation

$$\frac{\partial^2 \chi}{\partial x^2}+\frac{\partial^2 \chi}{\partial y^2}=0, \qquad \ldots \qquad \ldots \quad (2)$$

at all points of a cross section, and the condition

$$\frac{\partial \chi}{\partial \nu}=-l\{\tfrac{1}{2}\sigma x^2+(1-\tfrac{1}{2}\sigma)y^2\}-m(2+\sigma)xy, \qquad \ldots \quad (3)$$

l, m, 0 being the direction cosines of the outward drawn normal ν

The twist τ is to be so adjusted that the couple about the axis of z vanishes.

The strained central line cuts the strained cross-sections at the same angle $\frac{\pi}{2}-s_0$, where *

$$s_0=-\frac{W}{EI}\left(\frac{\partial \chi}{\partial x}\right)_0, \qquad \ldots \quad (4)$$

$\left(\frac{\partial \chi}{\partial x}\right)_0$ representing the value of $\frac{\partial \chi}{\partial x}$ at the centroid of the cross-section.

Semi elliptic Beam bounded by the Minor Axis, with the Minor Axis horizontal

The centroid of a cross-section lies on the major axis at a depth $ka=\frac{4a}{3\pi}$ below the minor axis, where a, b are the semi-axes of the elliptic boundary.

Let

$$x+ka=c \cosh \xi \cos \eta, \; y=c \sinh \xi \sin \eta \qquad \ldots \quad (5)$$

* Love, *loc cit.*, p 339

The curves ξ=constant, are ellipses with semi-axes $c\cosh\xi$, $c\sinh\xi$.

The curves η=constant, are confocal hyperbolas. The curve $\eta=\frac{\pi}{2}$, is the positive half of the y-axis, and the curve $\eta=-\frac{\pi}{2}$, is the negative half of the y-axis.

Also

$$\frac{1}{h^2}=\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \eta}\right)^2$$

$$=\frac{c^2}{2}(\cosh 2\xi-\cos 2\eta). \qquad \ldots \quad (6)$$

Let the boundaries be $\xi=\alpha$, and $\eta=\pm\frac{\pi}{2}$, so that

$$a=c\cosh\alpha,\ b=c\sinh\alpha. \qquad \ldots \quad (7)$$

On $\xi=\alpha$, we have

$$h\frac{\partial \chi}{\partial \xi}=-\frac{p(x+ka)}{a^2}\{\tfrac{1}{2}\sigma x^2+(1-\tfrac{1}{2}\sigma)y^2\}-\frac{py}{b^2}(2+\sigma)xy,$$

where $p=hab$.

This reduces to

$$\frac{\partial \chi}{\partial \xi}=a_1+b_1\cos\eta+c_1\cos 3\eta+d_1\cos 2\eta, \qquad \ldots \quad (8)$$

where,

$$\left.\begin{aligned}
a_1&=(1+\sigma)ka^2b,\\
b_1&=-(\tfrac{1}{4}+\tfrac{3}{8}\sigma)a^2b-\tfrac{1}{2}\sigma k^2a^2b-(\tfrac{3}{4}-\tfrac{1}{8}\sigma)b^3\\
c_1&=(\tfrac{1}{4}+\tfrac{1}{8}\sigma)a^2b+(\tfrac{1}{4}-\tfrac{1}{8}\sigma)b^3\\
d_1&=-ka^2b
\end{aligned}\right\} \qquad \ldots \quad (9)$$

On the minor axis, $x=-ka$, we have

$$\frac{\partial \chi}{\partial \nu}=-\frac{\partial \chi}{\partial x},\ l=-1,\ m=0,$$

so that

$$\frac{\partial \chi}{\partial x}=-\tfrac{1}{2}\sigma k^2a^2-(1-\tfrac{1}{2}\sigma)y^2. \qquad \ldots \quad (10)$$

If we assume

$$\chi=\chi_0=-k^2a^2x+(\tfrac{1}{3}-\tfrac{1}{6}\sigma)(x^3-3xy^2), \qquad \ldots \quad (11)$$

χ_0 satisfies the equation (2), and also the condition (3), on the minor axis $x=-ka$. But, on the boundary $\xi=a$, we have

$$\frac{\partial\chi_0}{\partial\xi}=b_2\cos\eta+c_2\cos 3\eta+d_2\cos 2\eta, \qquad \ldots \quad (12)$$

where

$$\left.\begin{aligned} b_2&=(\tfrac{1}{4}-\tfrac{1}{8}\sigma)a^2b-\tfrac{1}{2}\sigma k^2a^2b-(\tfrac{1}{4}-\tfrac{1}{8}\sigma)b^3,\\ c_2&=(\tfrac{3}{4}-\tfrac{3}{8}\sigma)a^2b+(\tfrac{1}{4}-\tfrac{1}{8}\sigma)b^3,\\ d_2&=-(2-\sigma)ka^2b. \end{aligned}\right\} \qquad \ldots \quad (13)$$

Hence, to satisfy the boundary condition (8), we take

$$\chi=\chi_0+\chi_1, \qquad \ldots \quad (14)$$

where χ_1 is a solution of (2) and is such that on the boundary $\xi=a$,

$$\frac{\partial\chi_1}{\partial\xi}=a_1+(b_1-b_2)\cos\eta+(c_1-c_2)\cos 3\eta+(d_1-d_2)\cos 2\eta$$

$$=a_3+b_3\cos\eta+c_3\cos 3\eta+d_3\cos 2\eta, \qquad \ldots \quad (15)$$

where

$$\left.\begin{aligned} a_3&=a_1=(1+\sigma)ka^2b,\\ b_3&=b_1-b_2=-(\tfrac{3}{4}+\tfrac{1}{2}\sigma)a^2b,\\ c_3&=c_1-c_2=(-\tfrac{1}{4}+\tfrac{1}{2}\sigma)a^2b,\\ d_3&=d_1-d_2=(1-\sigma)ka^2b, \end{aligned}\right\} \qquad \ldots \quad (16)$$

and on the boundaries $\eta=\pm\frac{\pi}{2}$, $\dfrac{\partial\chi_1}{\partial\eta}=0$. $\qquad \ldots \quad (17)$

Also $\dfrac{\partial\chi_1}{\partial\xi}$ should be continuous when $\xi=0$.

Expanding the right-hand side of (15), in Fourier's series of cosines of multiples of 2η, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, we have on the boundary $\xi=a$,

$$\frac{\partial\chi_1}{\partial\xi}=\sum_{n=1}^{\infty}A_{2n}\cos 2n\eta, \qquad \ldots \quad (18)$$

The constant term in the expansion is found to be zero and

$$A_{2n}=-\frac{4}{\pi}\cdot\frac{(-1)^n}{4n^2-1}b_2+\frac{12}{\pi}\cdot\frac{(-1)^n}{4n^2-9}c_2+d, \quad \ldots \quad (19)$$

where $d=d_2$ when $n=1$ and $d=0$ for all other values of n.

Assuming

$$\chi_1=\sum_{n=1}^{\infty} B_{2n}\cosh 2n\xi\cos 2n\eta, \quad \ldots \quad (20)$$

we find that it satisfies all the conditions of the problem, provided that

$$2nB_{2n}\sinh 2n\alpha=A_{2n} \quad \ldots \quad (21)$$

From symmetry, it is obvious that the twist is zero.

To find the obliquity of the strained central line to the cross-section, we observe that

$$\left(\frac{\partial\chi_0}{\partial x}\right)_0=-k^2a^2.$$

When $ka<c$ we have at the centroid, $\xi=0$, $\eta=\eta_0$, where $ka=c\cos\eta_0$ and then

$$\left(\frac{\partial\chi_1}{\partial x}\right)_0=\left(-h\frac{\partial\chi_1}{\partial\eta}\right)_{\xi=0,\ \eta=\eta_0}$$

$$=\frac{1}{c\sin\eta_0}\sum_1^{\infty} 2nB_{2n}\sin 2n\eta_0.$$

Therefore

$$s_0=\frac{W}{EI}\left[k^2a^2-\frac{1}{c\sin\eta_0}\sum_1^{\infty}2nB_{2n}\sin 2n\eta_0\right]. \quad \ldots \quad (22)$$

When $ka>c$, we have at the centroid, $\xi=\xi_0$, $\eta=0$, where $ka=c\cosh\xi_0$ and then

$$\left(\frac{\partial\chi_1}{\partial x}\right)_0=\left(h\frac{\partial\chi_1}{\partial\xi}\right)_{\xi=\xi_0,\eta=0}$$

$$=\frac{1}{c\sinh\xi_0}\sum_1^{\infty}2nB_{2n}\sinh 2n\xi_0.$$

Therefore

$$s_0=\frac{W}{EI}\left[k^2a^2-\frac{1}{c\sinh\xi_0}\sum_1^{\infty}2nB_{2n}\sinh 2n\xi_0\right]. \quad \ldots \quad (23)$$

Beam of Elliptic Section with two Symmetrical Keyways whose Boundaries are Confocal Hyperbolas

The major axis of the ellipse is taken to be horizontal

Let

$$x=c \sinh \xi \cos \eta,\ y=c \cosh \xi \sin \eta, \qquad \dots \quad (24)$$

so that

$$\frac{1}{h^2}=\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial x}{\partial \eta}\right)^2$$

$$=\frac{c^2}{2}(\cosh 2\xi+\cos 2\eta). \qquad \dots \quad (25)$$

ξ may have any value between $-\infty$ and $+\infty$, and η may have any value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. $\eta=0$ gives the x-axis, $\eta=-\frac{\pi}{2}$, the part of the y-axis between the left focus and $-\infty$ and $\eta=\frac{\pi}{2}$ the part of the y-axis between the right focus and ∞

Let the boundaries of a cross-section be

$$\xi=\pm\alpha,\ \eta=\pm\beta$$

For an elliptic section of semi axes a, b, with the b-axis vertical, χ is given by

$$\chi=\chi_0=-\frac{b^2\{2(1+\sigma)b^2+a^2\}}{3b^2+a^2}\,x+\frac{1}{3}\,\frac{2b^2+a^2+\frac{1}{2}\sigma(b^2-a^2)}{3b^2+a^2}\,(x^3-3xy^2) \quad (26)$$

Since $a=c \cosh \alpha$, $b=c \sinh \alpha$, we have

$$\chi_0=a_1 \sinh \xi \cos \eta+b_1 \sinh 3\xi \cos 3\eta, \qquad \dots \quad (27)$$

where

$$\left.\begin{aligned} a_1 \cosh \alpha &=-(\tfrac{1}{4}-\tfrac{1}{8}\sigma)a^3-(\tfrac{1}{2}+\tfrac{5}{8}\sigma)ab^2, \\ b_1 \cosh 3\alpha &=\tfrac{1}{3}[(\tfrac{1}{4}-\tfrac{1}{8}\sigma)a^3+(\tfrac{1}{2}+\tfrac{1}{8}\sigma)ab^2]. \end{aligned}\right\} \qquad \dots \quad (28)$$

Let us assume that

$$\chi=\chi_0+\chi_1, \qquad \dots \quad (29)$$

where χ_1 is a solution of (2).

Then we must have on the ellipse $\xi=\pm\alpha$,

$$\frac{\partial \chi_1}{\partial \nu}=0,\ i.e.,\ \frac{\partial \chi_1}{\partial \xi}=0 \qquad \dots \quad (30)$$

On the hyperbolas, $\eta=\pm\beta$,

$$\frac{\partial\chi}{\partial\nu}=-l\{\tfrac{1}{2}\sigma x^2+(1-\tfrac{1}{2}\sigma)y^2\}-m(2+\sigma)xy$$

But on the hyperbola, $\eta=\beta$,

$$l=-hc\sin\beta\sinh\xi,\ m=hc\cos\beta\cosh\xi,$$

and
$$\frac{\partial\chi}{\partial\nu}=h\frac{\partial\chi}{\partial\eta},$$

and on the hyperbola, $\eta=-\beta$,

$$l=-hc\sin\beta\sinh\xi,\ m=-hc\cos\beta\cosh\xi,$$

and
$$\frac{\partial\chi}{\partial\nu}=-h\frac{\partial\chi}{\partial\eta}.$$

Hence on $\eta=\beta$,

$$\frac{\partial\chi}{\partial\eta}=a_2\sinh\xi+b_2\sinh 3\xi, \quad \text{.. (31)}$$

and on $\eta=-\beta$,

$$-\frac{\partial\chi}{\partial\eta}=a_2\sinh\xi+b_2\sinh 3\xi, \quad \text{... (32)}$$

where

$$\left.\begin{aligned} a_2&=(\tfrac{1}{4}-\tfrac{1}{8}\sigma)c^3\sin^3\beta-(\tfrac{1}{4}+\tfrac{5}{8}\sigma)c^3\sin\beta\cos^2\beta,\\ b_2&=(\tfrac{1}{4}-\tfrac{1}{8}\sigma)c^3\sin^3\beta-(\tfrac{1}{4}+\tfrac{1}{8}\sigma)c^3\sin\beta\cos^2\beta,\end{aligned}\right\} \quad \text{... (33)}$$

When $\eta=\beta$,

$$\frac{\partial\chi_0}{\partial\eta}=-a_1\sinh\xi\sin\beta-3b_1\sinh 3\xi\sin 3\beta, \quad \text{... (34)}$$

and when $\eta=-\beta$,

$$\frac{\partial\chi_0}{\partial\eta}=a_1\sinh\xi\sin\beta+3b_1\sinh 3\xi\sin 3\beta. \quad \text{... (35)}$$

Therefore, when $\eta=\beta$,

$$\frac{\partial\chi_1}{\partial\eta}=a_3\sinh\xi+b_3\sinh 3\xi, \quad \text{... (36)}$$

and when $\eta=-\beta$,

$$-\frac{\partial\chi_1}{\partial\eta}=a_3\sinh\xi+b_3\sinh 3\xi, \quad \text{... (37)}$$

where

$$\left.\begin{aligned} a_3&=a_2+a_1\sin\beta\\ b_3&=b_2+3b_1\sin\beta\end{aligned}\right\} \quad \text{... (38)}$$

Since $\frac{\partial \chi_1}{\partial \xi} \equiv 0$ when $\xi = \pm a$, we expand $\frac{\partial \chi_1}{\partial \eta}$, in a Fourier's series of sines of odd multiples of $\frac{\pi\xi}{2a}$.

Hence when $\eta = \beta$,

$$\frac{\partial \chi_1}{\partial \eta} = \sum_{n=0}^{\infty} A_{2n+1} \sin \frac{(2n+1)\pi\xi}{2a}, \qquad \ldots \quad (39)$$

and when $\eta = -\beta$,

$$-\frac{\partial \chi_1}{\partial \eta} = \sum_{n=0}^{\infty} A_{2n+1} \sin \frac{(2n+1)\pi\xi}{2a}, \qquad \ldots \quad (40)$$

where

$$A_{2n+1} = (-1)^n \frac{8a \cosh a}{(2n+1)^2\pi^2 + 4a^2} a_1 + (-1)^n \frac{24a \cosh 3a}{(2n+1)^2\pi^2 + 36a^2} b_3 \quad (41)$$

Hence

$$\chi_1 = \sum_{n=0}^{\infty} B_{2n+1} \cosh \frac{(2n+1)\pi\eta}{2a} \sin \frac{(2n+1)\pi\xi}{2a}, \qquad \ldots \quad (42)$$

where

$$B_{2n+1} \sinh \frac{(2n+1)\pi\beta}{2a} = \frac{2a}{(2n+1)\pi} A_{2n+1} \qquad \ldots \quad (43)$$

From symmetry, it is obvious that the twist is zero.

To obtain the obliquity of the strained central line to the strained cross-sections, we observe that

$$\left(\frac{\partial \chi_0}{\partial x}\right)_0 = -\frac{b^2\{2(1+\sigma)b^2 + a^2\}}{3b^2 + a^2},$$

and

$$\left(\frac{\partial \chi_1}{\partial x}\right)_0 = \left(h\frac{\partial \chi_1}{\partial \xi}\right)_0 = \frac{1}{c} \sum_{n=0}^{\infty} \frac{(2n+1)\pi}{2a} B_{2n+1}.$$

Therefore

$$s_0 = \frac{W}{EI}\left[\frac{b^2\{2(1+\sigma)b^2 + a^2\}}{3b^2 + a^2} - \frac{1}{c} \sum_{n=0}^{\infty} \frac{(2n+1)\pi}{2a} B_{2n+1}\right] \qquad \ldots \quad (44)$$

Putting $\beta = \frac{\pi}{2}$, we get the case of the flexure of an elliptic beam, with two slits extending from the foci to the ends of the major axis.

Bull. Cal. Math. Soc., Vol. XXVII, Nos. 1 & 2 (1935).

8

A Theorem on the Non-Euclidean Triangle

BY

R. C. Bose

The object of this short paper is to prove an interesting theorem giving the relation between the sides and altitudes of a Non-Euclidean triangle and to deduce from it a synthetic proof of the Median Theorem.

I.

Given a pair of segments x, y we can obtain from them an angle $\phi(x, y)$ in the following manner. Let AOB be a right-angle in which $OA=x$, $OB=y$ Then $\phi(x, y)$ is the angle between the lines perpendicular to OA and OB at A and B respectively. If u, v be another pair of segments and the angle $\phi(u, v)$ obtained from them in a similar manner be congruent to $\phi(x, y)$ the relation between the two pairs of segments will be denoted by writing

$$\phi(x, y)=\phi(u, v)$$

In Elliptic Geometry the angle between any two lines is always *actual*. In Hyperbolic Geometry there exist *null* and *ideal* angles also. A null angle is the angle between a pair of parallel lines. All null angles are congruent. An ideal angle is an angle between two ultra parallel straight lines. Two ideal angles are to be regarded as congruent if the distance between the arms of the one (measured along the common perpendicular to the arms) is equal to the distance between the arms of the other.

It is clear from the definition of $\phi(x, y)$ that if $\phi(x, y)=\phi(u, v)$ and $x=u$ then $y=v$. Also $\phi(x, y)=\phi(y, x)$.

II

We can now state the following theorem for the Elliptic or the Hyperbolic Geometry.

Theorem. *If p, q, r are the altitudes of a triangle ABC corresponding to the sides a, b, c*

$$\phi(a, p) = \phi(b, q) = \phi(c, r)$$

Let D, E, F be the feet of the perpendiculars from A, B, C to the opposite sides so that $BC=a$, $CA=b$, $AB=c$, $AD=p$, $BE=q$, $CF=r$ (see Fig. i.)

We proceed in the first instance to prove the theorem for the case of the Elliptic Geometry.

Let the perpendiculars to BE and CF at B and C respectively meet in O_1. Produce O_1B to O_3, O_1C to O_2 making $O_1B=BO_3$, $O_1C=CO_2$. Drop O_iP_i, O_iQ_i, O_iR_i perpendiculars from O_i to BC, CA and AB respectively ($i=1, 2, 3$) (see Fig. ii).

It follows from the congruence of the triangles O_1Q_1C and O_2Q_2C that $O_1Q_1=O_2Q_2$. Again from the congruence of the quadrilaterals O_1BEQ_1 and O_3BEQ_3, $O_1Q_1=O_3Q_3$. Hence $O_2Q_2=O_3Q_3$ and the line O_2O_3 must be bisected at the mid-point of Q_2Q_3. In the same way we show that O_2O_3 must be bisected at the mid-point of R_2R_3. But the segments Q_2Q_3 and R_2R_3 have only the point A in common. It is thus clear that O_2O_3 is bisected at A.

Again $O_2P_2=O_3P_3$ each being equal to O_1P_1. Hence the quadrilaterals O_2ADP_2 and O_3ADP_3 are congruent and the angle O_2AD is right. Also $P_2D=P_3D$.

Thus $O_1O_2O_3$ is a triangle of which AD, BE and CF are right-bisectors.

Now $P_1B=P_3B$ and $P_1C=P_2C$. Therefore $P_2P_3=2BC$ or

$$P_2D=P_3D=a.$$

Hence by definition

$$\phi(a, p) = \angle AO_2P_2 = \angle AO_3P_3$$

But it is easy to see that

$$\angle AO_2P_2 + \angle AO_3P_3 = \tfrac{1}{2}(\lambda+\mu+\nu)$$

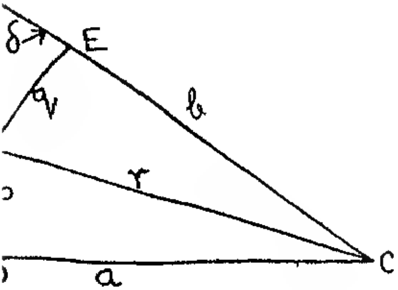


Fig i

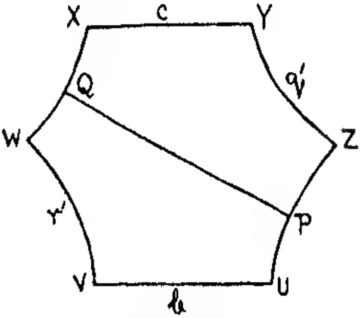
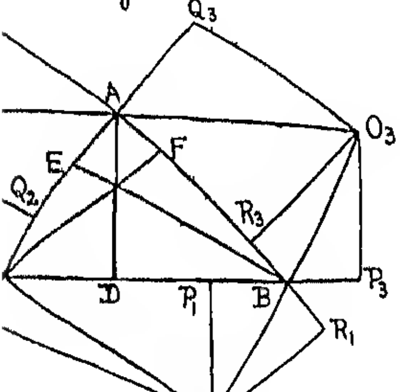


Fig iii



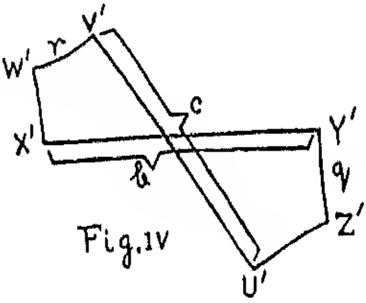


Fig. iv

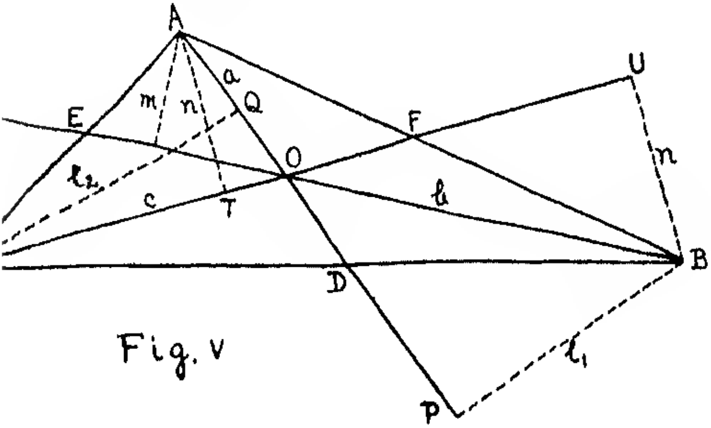


Fig. v

where λ, μ, ν are the angles of the triangle $O_1O_2O_3$. Thus

$$\phi(a, p)=\tfrac{1}{2}(\lambda+\mu+\nu)$$

By symmetry $\phi(b, q)$ and $\phi(c, r)$ are also each equal to $\frac{1}{2}(\lambda+\mu+\nu)$. Therefore

$$\phi(a, p)=\phi(b, q)=\phi(c, r)$$

III

In the case of the Hyperbolic Geometry the perpendiculars to BE and CF at B and C respectively, may either intersect, be parallel or be ultra parallel and thus possess a common perpendicular. In the first case the proof of the previous paragraph is still valid. Suitable modifications of the proof suffice to cover the other two cases.

An elegant proof applying to all possible cases can however be obtained, by using the correspondence between rectangular hexagons on the Hyperbolic Plane, enunciated by the writer in a previous issue of this bulletin.*

Denote the angle CAB by δ (see Fig. i). Then corresponding to the right angled triangle ABE in which the hypotenuse $AB=c$, the side $BE=q$ and the $\angle BAE=\delta$ there exists a rectangular pentagon† XYZPQ in which $XY=c$, $YZ=q'$, $PQ=d$ where q' is the segment complementary to q and d is the parallel distance corresponding to δ. Similarly corresponding to the right-angled triangle ACF there exists the rectangular pentagon UVWQP in which $UV=b$, $VW=r'$, $QP=d$ where r' is the segment complementary to r. Putting together the two pentagons as in Fig. iii we obtain a rectangular hexagon XYZUVW for which $XY=c$, $YZ=q'$, $UV=b$, $VW=r'$.

Corresponding to this we must have a crossed rectangular hexagon $X'Y'Z'U'V'W'$ (see Fig. iv) for which $X'Y'=b$, $Y'Z'=q$, $U'V'=c$, $V'W'=r$. By definition each of the angles $\phi(b, q)$ and $\phi(c, r)$ is given by the angle between the lines $Z'U'$ and $W'X'$.

N.B. The theorem proved above for the Non Euclidean Geometries remains true for Euclidean Geometry provided that by $\phi(x, y)$ we

* R. C. Bose, "Theory of Associated Figures in Hyperbolic Geometry," Bull. Cal. Math. Soc., Vol. xix, 1928, Th. III, p. 118.

† S. Mukhopadhyaya, "Geometrical investigations on the correspondences between a right angled triangle, a three-right angled quadrilateral and a rectangular petagon in Hyperbolic Geometry," Bull. Cal. Math. Soc., Vol. xiii, 1922-23, p. 215.
Also R. C. Bose, Loc. cit., Th. II, p. 108.

understand in this case the content* of the rectangle whose sides are x and y

The following proof of the median theorem then applies to each of the three standard geometries

IV.

Theorem. If D, E, F *be the mid-points of the sides* BC, CA, AB *of a triangle* ABC, *the lines* AD, BE, CF *are concurrent.*

Let BE and CF meet in O. Join AO Draw BP, CQ perpendicular to AO, AS, CR perpendicular to BO and AT, BU perpendicular to CO (see Fig. v) Set

$$AO=a,\ BO=b,\ CO=c.\ BP=l_1,\ CQ=l_2$$

$$AS=CR=m,\ BU=AT=n.$$

Then $$\phi(a, l_1)=\phi(b, m) \text{ from } \triangle AOB$$

$$=\phi(c, n) \text{ from } \triangle BOC$$

$$=\phi(a, l_2) \text{ from } \triangle COA$$

Hence $l_1=l_2$ or AO passes through D

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* *Cf* "Foundations of Geometry" by Hilbert English Trans by Townsend Second Edition, p. 58.

9

Note on the application of trilinear co-ordinates in some problems of Elasticity and Hydrodynamics

BY

Bibhutibhushan Sen

1. *Introduction.*

In a previous communication to this Bulletin,* it has been shown that the use of trilinear co-ordinates simplifies the solution of several problems of elasticity connected with thin plates having equilateral triangles as boundaries. In this note, two more problems are solved, namely, the problem of the bending of an equilateral triangular plate supported on *flexible beams* and that of the oscillation of water in a basin having for its section an equilateral triangle.

For defining the trilinear co-ordinates, we take O, the incentre of the equilateral triangle ABC as the origin and lines parallel and perpendicular to the side BC as the axes of y and x respectively.

Let (x, y) be the cartesian co-ordinates of a point P of which the distances from the sides CA, AB and BC are respectively p_1, p_2 and p_3. Then if r be the radius of the inscribed circle and $2a$ the length of each side, we have

$$p_1 = r + \frac{x}{2} - \frac{y\sqrt{3}}{2},$$

$$p_2 = r + \frac{x}{2} + \frac{y\sqrt{3}}{2},$$

$$p_3 = r - x. \tag{1.1}$$

* *Vide* Vol. 20, p. 65.

Hence

$$p_1+p_2+p_3=3r=a\sqrt{3}=k \text{ (say)}. \tag{1.2}$$

2. *An equilateral triangular plate supported on flexible beams.*

Let Z be the distributed load per unit of area and D the flexural rigidity of the plate Then the normal displacement w satisfies the equation*

$$\nabla_1^4 w=\frac{Z}{D}=Z_0 \quad \text{(say)} \tag{2.1}$$

If the bounding lines†

$$p_1=0,\ p_2=0 \text{ and } p_3=0$$

define the positions of the supporting beams and the points A, B and C the positions of the vertical columns to which the horizontal beams are attached, we have the following boundary conditions.

$$\text{When } p_1=0, \frac{\partial w}{\partial \nu}=-\frac{\partial w}{\partial p_1}+\frac{1}{2}\frac{\partial w}{\partial p_2}+\frac{1}{2}\frac{\partial w}{\partial p_3}=0\,; \tag{2.2}$$

$$\text{when } p_2=0, \frac{\partial w}{\partial \nu}=-\frac{\partial w}{\partial p_2}+\frac{1}{2}\frac{\partial w}{\partial p_3}+\frac{1}{2}\frac{\partial w}{\partial p_1}=0\,, \tag{2.3}$$

$$\text{when } p_3=0, \frac{\partial w}{\partial \nu}=-\frac{\partial w}{\partial p_3}+\frac{1}{2}\frac{\partial w}{\partial p_1}+\frac{1}{2}\frac{\partial w}{\partial p_2}=0\,, \tag{2.4}$$

and

$$w=0 \text{ at A, where } p_1=p_2=0 \text{ and } p_3=h, \tag{2.5}$$

$$w=0 \text{ at B, where } p_2=p_3=0 \text{ and } p_1=h, \tag{2.6}$$

$$w=0 \text{ at C, where } p_1=p_3=0 \text{ and } p_2=h \tag{2.7}$$

* "The Mathematical Theory of Elasticity" by A. E. H Love, 4th Edition, p. 488.

† Vide "Rectangular plates on flexible beams" by E H. Bateman published in the Philosophical Magazine, ser 7, Vol 20, (1935), p. 607.

The equation (2 1) in terms of p_1, p_2 and p_3 stands as

$$\left[\frac{\partial^2}{\partial p_1^2}+\frac{\partial^2}{\partial p_2^2}+\frac{\partial^2}{\partial p_3^2}-\frac{\partial^2}{\partial p_1 \partial p_2}-\frac{\partial^2}{\partial p_1 \partial p_3}-\frac{\partial^2}{\partial p_2 \partial p_3}\right]^2 w=Z_0 \quad (2\,8)$$

As a solution of this equation let us write

$$w=P(p_1^4+p_2^4+p_3^4)+Qp_1p_2p_3+R(p_1^2+p_2^2+p_3^2)+S, \quad \ldots \quad (2\,9)$$

where P, Q, R, S are constants

Then we find that the equation is satisfied if

$$P=\frac{Z_0}{72} \quad . \quad \ldots \quad (2\,10)$$

On the side $p_1=0$

$$-\frac{\partial w}{\partial p_1}+\tfrac{1}{2}\,\frac{\partial w}{\partial p_2}+\tfrac{1}{2}\,\frac{\partial w}{\partial p_3}$$

$$=-Qp_2p_3+2P(p_2^3+p_3^3)+R(p_2+p_3).$$

Since on this boundary

$$p_2+p_3=h,$$

the above expression becomes

$$-Qp_2p_3+2P(h^3-3hp_2p_3)+Rh$$

Hence it will be zero if we take

$$Q=-\frac{Z_0 h}{12}\text{, and} \quad (2,11)$$

$$R=-\frac{Z_0 h^2}{36}\ . \quad (2,12)$$

As only the symmetrical functions of p_1, p_2 and p_3 are involved in the expression for w given in (2 9), it is evident that the boundary

conditions (2.3) and (2.4) are also satisfied for these values of Q and R. Again putting

$p_1=0$, $p_2=0$ and $p_3=k$ in (2.9), we obtain the condition (2.5) as

$$Pk^4+Rk^2+S=0,$$

whence we get

$$S=\frac{Z_0 k^4}{72}. \qquad \ldots \quad (2.13)$$

For this value of S, the conditions (2.6) and (2.7) are also satisfied.

Hence the required value of

$$w=\frac{Z_0}{72}\left[(p_1^4+p_2^4+p_3^4)-6kp_1p_2p_3-2k^2(p_1^2+p_2^2+p_3^2)+k^4\right]. \qquad (2.14)$$

At the origin

$$p_1=p_2=p_3=r,$$

and there the deflection

$$w=\frac{Zr^4}{6D}. \qquad \ldots \quad (2.15)$$

3. *Oscillation of water in a basin having an equilateral triangle for its section.*

For finding the free oscillation of a sheet of water bounded by vertical walls of height h, we require the solution of the equation*

$$(\nabla_1^2+\lambda^2)\zeta=0, \qquad \ldots \quad (3.1)$$

subject to the boundary condition

$$\frac{\partial\zeta}{\partial n}=0 \qquad \ldots \quad (3.2)$$

where δn denotes an element of the normal to the boundary, ζ denotes the elevation of the free surface above the undisturbed level, and $\dfrac{2\pi}{\lambda\sqrt{gh}}$ denotes the period of a normal mode of oscillation to be determined.

* Lamb's Hydrodynamics, 4th Edition, p. 276

The equation (3 1) expressed in terms of p_1, p_2, p_3 becomes

$$\left[\frac{\partial^2}{\partial p_1^2}+\frac{\partial^2}{\partial p_2^2}+\frac{\partial^2}{\partial p_3^2}-\frac{\partial^2}{\partial p_1 \partial p_2}-\frac{\partial^2}{\partial p_1 \partial p_3}-\frac{\partial^2}{\partial p_2 \partial p_3}+\lambda^2\right]\zeta=0, \quad (3\ 3)$$

while the boundary condition (3 2) is equivalent to

$$\left(-\frac{\partial}{\partial p_1}+\tfrac{1}{2}\frac{\partial}{\partial p_2}+\tfrac{1}{2}\frac{\partial}{\partial p_3}\right)\zeta=0 \text{ when } p_1=0, \quad \ldots \quad (3\ 4)$$

$$\left(-\frac{\partial}{\partial p_2}+\tfrac{1}{2}\frac{\partial}{\partial p_3}+\tfrac{1}{2}\frac{\partial}{\partial p_1}\right)\zeta=0 \text{ when } p_2=0, \text{ and} \quad \ldots \quad (3\ 5)$$

$$\left(-\frac{\partial}{\partial p_3}+\tfrac{1}{2}\frac{\partial}{\partial p_1}+\tfrac{1}{2}\frac{\partial}{\partial p_2}\right)\zeta=0 \text{ when } p_3=0 \quad \ldots \quad (3\ 6)$$

For a simple symmetrical mode of oscillation, let us assume

$$\zeta=\Lambda_m\left[\cos\frac{2m\pi p_1}{k}+\cos\frac{2m\pi p_2}{k}+\cos\frac{2m\pi p_3}{k}\right] \quad \ldots \quad (3\ 7)$$

where m is an integer and Λ_m a constant

Then

$$\left[-\frac{\partial}{\partial p_1}+\tfrac{1}{2}\frac{\partial}{\partial p_2}+\tfrac{1}{2}\frac{\partial}{\partial p_3}\right]\zeta$$

$$=\frac{2m\pi}{k}\Lambda_m\left[\sin\frac{2m\pi p_1}{k}-\sin\frac{m\pi(p_2+p_3)}{k}\cos\frac{m\pi(p_2-p_3)}{k}\right]$$

$$=0$$

when $p_1=0$ and $p_2+p_3=k$.

Similarly it can be shown that the other boundary conditions are also satisfied.

The equation (3.3) is satisfied if

$$\frac{4m^2\pi^2}{k^2}=\lambda^2, \qquad \ldots \quad (3\ 8)$$

For different integral values of m, different values of periods are obtained from the above relation When $m=1$, we have the longest period

$$=\frac{2\pi}{\frac{2\pi}{k}\sqrt{gh}}=\sqrt{\frac{3}{gh}}\,a,$$

$2a$ being the length of either side

10

Some Properties of the Convex Oval with Reference to its Perimeter Centroid

BY

R. C. Bose and S. N. Roy

(*Calcutta University*)

1. *Introduction*

Steiner* has defined the *curvature centroid* of an oval as the centre of mass of the perimeter of the oval, when every point of the perimeter is considered to have a density equal to the curvature at that point. Hayashi has investigated the properties of a convex oval with reference to the curvature centroid.† Another important point connected with the oval is the centroid of the perimeter which we call the *perimeter centroid.* Meissner has shown that *the perimeter centroid of an oval of constant breadth coincides with its curvature centroid,*‡ and Kubota has proved the elegant theorem that *the locus of the perimeter centroid for a system of parallel ovals is a straight line* § We here deduce the co-ordinates of the perimeter centroid, when the tangential polar

* J. Steiner. Von dem Krümmungsschwerpunkte ebener Kurven. Crelle J. 21 (1838).

† T. Hayashi: Rend. Circ. Matem. Palermo, t. L. (1926), pp 96-102

‡ Meissner, Über die Anwendung von Fourier Reihen auf einige aufgaben der Geometrie und Kinematik, Vierteljahrschrift der Naturforschenden Gessellschaft in Zurich 54 (1909) Also F. Schilling, Die Theorie u. Konstruktion der Kurve Konstanter Breite. Zeitschrift fur Math. u. Physik, 1914

§ T. Kubota. Über die Schwerpunkte der convexen geschlossenen Kurven und Flächen, Tohoku Math J., Vol. 14 (1918), pp 20-27.

equation of the oval is supposed to be known, and use this result to obtain a new proof of the theorems of Meissner and Kubota. We then go on to obtain properties of the convex oval with reference to the perimeter centroid, which are analogous to the properties with reference to the curvature centroid studied by Hayashi. We thus prove :—

(*a*) *If p denote the length of the perpendicular on the tangent, from the perimeter centroid, and r denotes the radius vector to the point of contact then $3p^2-r^2$ takes the value R^2 at least four times, where R is the radius of a circle, whose area is equal to the sum of the areas of the oval and its pedal with respect to its perimeter centroid.*

(*b*) *If n denote the number of normals which can be drawn from the perimeter centroid to the oval, and m denote the number of points for which $\rho=3p$, ρ being the radius of curvature and p the perpendicular from the perimeter centroid to the tangent, then $m+n\geqq 4$.*

2. *Co-ordinates of the perimeter centroid when the tangential polar equation of the oval is given*

If the positive tangent at any point (x, y) of the oval, makes an angle ψ, with the positive direction of the axis of x, and if p denotes the length of the perpendicular drawn from the origin to the tangent, then the equation of the tangent can be written as

$$x \sin\psi - y\cos\psi = p$$

$$\therefore \qquad x\cos\psi + y\sin\psi = \frac{dp}{d\psi}$$

Hence
$$x = p\sin\psi + \cos\psi\,\frac{dp}{d\psi} \qquad \text{.. (1)}$$

$$y = -p\cos\psi + \sin\psi\,\frac{dp}{d\psi} \qquad \text{.. (2)}$$

If the tangential polar equation $p=f(\psi)$ of the oval is known, the relations (1) and (2) give the Cartesian co-ordinates of any point on the oval

If ρ denotes the radius of curvature at any point of the oval and dashes denote differentiations with respect to p, we know that

$$\rho = p + p'' \qquad \dots \quad (3)$$

Now let x_0, y_0 be the Cartesian co-ordinates of the perimeter centroid and let L_0 be the perimeter of the oval, so that

$$L_0 = \int_0^{2\pi} p\,d\psi \qquad \dots \quad (4)$$

$$L_0 x_0 = \int_0^{2\pi} x\,ds$$

$$= \int_0^{2\pi} x\rho\,d\psi$$

$$= \int_0^{2\pi} (p \sin\psi + p' \cos\psi)(p + p'')\,d\psi, \qquad \text{from (1) and (3)}$$

$$= \int_0^{2\pi} p^2 \sin\psi\,d\psi + \int_0^{2\pi} pp'' \sin\psi\,d\psi$$

$$+ \int_0^{2\pi} pp' \cos\psi\,d\psi + \int_0^{2\pi} p'p'' \cos\psi\,d\psi \qquad \dots \quad (5)$$

The second, third and fourth of these integrals can be evaluated by integration by parts and noticing that the parts outside the sign of integration always vanish between the limits 0 to 2π. Thus

$$\int_0^{2\pi} p'p'' \cos\psi\,d\psi = \tfrac{1}{2} \int_0^{2\pi} p'^2 \sin\psi\,d\psi \qquad \dots \quad (6)$$

$$\int_0^{2\pi} pp' \cos\psi\,d\psi = \tfrac{1}{2} \int_0^{2\pi} p^2 \sin\psi\,d\psi \qquad \dots \quad (7)$$

$$\int_0^{2\pi} pp'' \sin\psi\,d\psi = -\int p'(p \cos\psi + p' \sin\psi)\,d\psi$$

$$= -\int p'^2 \sin\psi\,d\psi - \tfrac{1}{2} \int p^2 \sin\psi\,d\psi. \qquad \dots \quad (8)$$

Substituting from (6), (7) and (8) in (5) we have

$$x_0 = \frac{1}{L_0}\int_0^{2\pi}(p^2-\tfrac{1}{2}p'^2)\sin\psi d\psi \quad \dots \quad (9)$$

Likewise we can show that

$$y_0 = -\frac{1}{L_0}\int_0^{2\pi}(p^2-\tfrac{1}{2}p'^2)\cos\psi d\psi \quad \dots \quad (10)$$

3. *Perimeter centroid for an oval of constant breadth*

For an oval of constant breadth we have the relation

$$p(\psi)+p(\psi+\pi)=b, \quad (11)$$

where b is the breadth

$$\therefore \quad p'(\psi)+p'(\psi+\pi)=0$$

From this it follows at once that

$$\int_0^{2\pi} p'^2 \sin\psi d\psi = 0 \quad \dots \quad (12)$$

$$\int_0^{2\pi} p'^2 \cos\psi d\psi = 0. \quad (13)$$

$$x_0 = \frac{1}{L_0}\int_0^{2\pi} p^2 \sin\psi d\psi \qquad \text{from (9) and (12)}$$

$$= \frac{1}{L_0}\int_0^{\pi} \{p^2-(b-p)^2\}\sin\psi d\psi \qquad \text{from (11)}$$

$$= \frac{2b}{L_0}\int_0^{\pi} p\sin\psi d\psi - \frac{2b^2}{L}$$

But from Barbier's theorem * $L=\pi b$

$$\therefore \quad x_0 = \frac{2}{\pi}\int_0^{\pi} p\sin\psi d\psi - \frac{2b}{\pi} \quad \dots \quad (14)$$

* Barbier Liouville's Journal (2) 5 (1860), pp 273-86.

Likewise

$$y_0 = -\frac{2}{\pi}\int_0^{\pi} p\cos\psi d\psi. \quad \ldots \quad (15)$$

But if $(\bar{x}, \bar{y})$ are the co-ordinates of the curvature centroid of any oval

$$\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} \frac{x}{\rho}\, ds = \frac{1}{\pi}\int_0^{2\pi} p\sin\psi d\psi \quad \ldots \quad (16)$$

$$\bar{y} = \frac{1}{2\pi}\int_0^{2\pi} \frac{y}{\rho}\, ds = -\frac{1}{\pi}\int_0^{2\pi} p\cos\psi d\psi \quad \ldots \quad (17)$$

When however the oval is of constant breadth, using the relation (11) we at once have

$$\bar{x} = \frac{2}{\pi}\int_0^{\pi} p\sin\psi d\psi - \frac{2b}{\pi} \quad \ldots \quad (18)$$

$$\bar{y} = -\frac{2}{\pi}\int_0^{\pi} p\cos\psi d\psi \quad \ldots \quad (19)$$

Comparing the results (14), (15), (18), (19) we see at once that

For an oval of constant breadth the perimeter centroid coincides with the curvature centroid

4. *Locus of the perimeter centroid for a series of equidistant ovals.*

If $p=f(\psi)$ is the tangential polar equation of any oval, then we can construct an equidistant oval by cutting off a distance ϵ along all outward normals and joining the points so obtained If we vary ϵ we get a series of equidistant ovals with equation $p+\epsilon=f(\psi)$ We shall take the curvature centroid of the original oval as our origin. It is easy to see that this will remain the curvature centroid of the whole series. With origin so chosen

$$\int_0^{2\pi} p\sin\psi d\psi = 0, \int_0^{2\pi} p\cos\psi d\psi = 0. \quad \ldots \quad (20)$$

Let L_ϵ denote the perimeter and x_ϵ, y_ϵ the co-ordinates of the perimeter centroid, of the oval of the series which is at a distance ϵ from the original oval. We then have from (9) and (20)

$$L_\epsilon x_\epsilon = \int_0^{2\pi} \{ (p+\epsilon)^2 - \tfrac{1}{2}p'^2 \} \sin \psi d\psi = L_0 x_0 \qquad \ldots \quad (21)$$

Similarly

$$L_\epsilon y_\epsilon = L_0 y_0 \qquad (22)$$

$$\therefore \qquad y_\epsilon / x_\epsilon = y_0 / x_0 = \text{const.} \qquad \ldots \quad (23)$$

Also

$$x_\epsilon^2 + y_\epsilon^2 = \frac{L_0^2}{L_\epsilon^2} (x_0^2 + y_0^2). \qquad \ldots \quad (24)$$

We can thus state :—

In a series of equidistant ovals, the locus of the perimeter centroid is a straight line passing through the curvature centroid, which remains fixed, while the distance between the two centroids varies inversely as the perimeter of the oval.

5. *Properties of the convex oval, with reference to its perimeter centroid.*

Hayashi has shown in the paper referred to in the introduction, that in virtue of Blaschke's mechanical proof of the four cyclic point theorem or from certain theorems of Hurwitz on Fourier series, it follows immediately that if $f(\psi)$ be a one-valued continuous periodic function with period 2π, satisfying the relations

$$\int_0^{2\pi} f(\psi) \sin \psi d\psi = 0, \quad \int_0^{2\pi} f(\psi) \cos \psi d\psi = 0 \qquad \ldots \quad (25)$$

then $f(\psi)$ has at least four extrema in the complete period, and takes on its mean value

$$\frac{1}{2\pi} \int_0^{2\pi} f(\psi) d\psi \qquad \ldots \quad (26)$$

at least four times

If we now take the perimeter centroid of our oval as the origin, and set

$$f(\psi)\equiv p^2-\tfrac{1}{2}p'^2=\tfrac{1}{2}(3p^2-r^2) \tag{27}$$

where r is the radius vector to the point of contact, then the formulae (9) and (10) show at once that $f(\psi)$ satisfies the relations (25) Hence $f(\psi)$ has four extrema in the complete period But

$$f'(\psi)=(2p-p'')p'=(3p-\rho)p'$$

Thus for an extremum of $f(\psi)$ either $\rho=3p$ or $p'=0$, *i e*, the normal passes through the origin. Hence

If n denotes the number of normals which can be drawn from the perimeter centroid to the oval, and if m denotes the number of points for which $\rho=3p$, where p is the perpendicular from the perimeter centroid to the tangent, then $m+n\geqq 4$

Again let A be the area of the oval, and B the area of the pedal of the oval with respect to its perimeter centroid. Then

$$\frac{1}{2\pi}\int_0^{2\pi} f(\psi)d\psi=\frac{1}{2\pi}\int_0^{2\pi}(p^2-\tfrac{1}{2}p'^2)d\psi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}p^2d\psi+\frac{1}{4\pi}\int_0^{2\pi}pp''d\psi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}p^2d\psi+\frac{1}{4\pi}\int_0^{2\pi}p(\rho-p)d\psi$$

$$=\frac{1}{4\pi}\int_0^{2\pi}p^2d\psi+\frac{1}{4\pi}\int_0^{2\pi}p\rho d\psi$$

$$=\frac{1}{2\pi}(\mathrm{A}+\mathrm{B}). \tag{28}$$

Thus $3p^2-r^2$ takes the value $(\mathrm{A}+\mathrm{B})/\pi$ at least four times We may express this by saying

If p denotes the length of the perpendicular from the perimeter centroid on the tangent and r denotes the radius vector to the point of contact, then $3p^2-r^2$ *takes the value* R^2 *at least four times, where R is the radius of a circle whose area is equal to the sum of the areas of the oval, and its pedal with respect to the perimeter centroid*

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11

Some Theorems on Geodesic Curvature and Geodesic Parallels

BY

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(Communicated by the Secretary)

1. In the Mathematical Gazette, Vol. 13 (1926), Dr C E Weatherburn proved some theorems regarding the Line of Striction of a family of geodesics. In the same paper it has been proved that "A curve drawn on a surface so as to cut a family of geodesics and possessing any two of the following properties :—(a) it is a geodesic (b) it is the line of striction of the family of geodesics (c) it cuts the family of geodesics at a constant angle, also possesses the third property" In the present paper attempt has been made to extend the properties to a family of curves cutting a family of geodesic parallels at a constant angle and also to derive some other theorems believed to be new.

2 Let $\bar{a}$ and $\bar{b}$ be unit tangents to two families of curves cutting at a constant angle α on a surface The unit tangent $\bar{t}$ to a family of curves cutting the family having $\bar{a}$ for its unit tangent at a constant angle θ is given by

$$\bar{t} = \frac{1}{\sin\alpha}\{\bar{a}\sin(\alpha-\theta)+\bar{b}\sin\theta\}$$

The unit tangent to the orthogonal trajectory is given by

$$\bar{t}' = \frac{1}{\sin\alpha}\{\bar{b}\cos\theta-\bar{a}\cos(\alpha-\theta)\}.$$

Now $-\operatorname{div}\bar{t}' = \frac{1}{\sin\alpha}\{\cos(\alpha-\theta)\operatorname{div}\alpha - \cos\theta\operatorname{div}\bar{b}$

$$+(\bar{a}\sin(\alpha-\theta)+\bar{b}\sin\theta)\nabla\theta\}$$

$$=\frac{1}{\sin\alpha}\{\cos(\alpha-\theta)\operatorname{div}\bar{a}-\cos\theta\operatorname{div}\bar{b}+\bar{t}\nabla\theta\}$$

$$=\frac{1}{\sin\alpha}\left\{\cos(\alpha-\theta)\operatorname{div}\bar{a}-\cos\theta\operatorname{div}\bar{b}+\frac{d\theta}{ds}\right\} \quad \ldots \quad (2\ 1)$$

And similarly

$$\operatorname{div}\bar{t} = \frac{1}{\sin\alpha}\{\sin(\alpha-\theta)\operatorname{div}\bar{a}+\sin\theta\operatorname{div}\bar{b}+t'\nabla\theta\} \quad \ldots \quad (2\ 2)$$

If therefore the family having $\bar{a}$ for its unit tangent be a family of parallels,

$$\operatorname{div}\bar{a}=0$$

and
$$-\operatorname{div}\bar{t}' = \frac{1}{\sin\alpha}\left\{-\cos\theta\operatorname{div}\bar{b}+\frac{d\theta}{ds}\right\}. \quad \ldots \quad (2\cdot 11)$$

The vanishing of any two of the quantities $\operatorname{div}\bar{t}'$, $\operatorname{div}\bar{b}$ and $\frac{d\theta}{ds}$, requires the vanishing of the third Hence we have the theorem "A curve drawn on a surface so as to cut a family of parallels and possessing two of the properties (*a*) it is a geodesic (*b*) it cuts the family of parallels at a constant angle (*c*) it is the line of striction of the oblique trajectories to the family of parallels, also possesses the third property.

Again from (2·2) it is evident that if $\operatorname{div}\alpha=0$ and $\frac{d\theta}{ds}=0$, $\operatorname{div}\ t$ and $\operatorname{div}\ b$ vanishes simultaneously Hence the theorem that "The line of striction of two families of oblique trajectories to a set of geodesic parallels are identical"

3 Voss's Theorem in Differential Geometry states that "If the geodesic curvature of an orthogonal family of curves on a surface be constant, the surface has a constant negative second curvature" If however the geodesic curvature of two families of curves cutting one another at a constant angle be constant along each member of the oblique trajectory, the surface has a negative second curvature but not necessarily constant.

Let the families of curves cutting at a constant angle α be taken as the parametric curve so that

$$ds^2 = E du^2 + 2\sqrt{EG}\ \cos\alpha\, du dv + G dv^2$$

The geodesic curvature of the family v=constant is given by

$$k_{gu} = H\lambda E^{-\frac{3}{2}}$$

where $$H = \sqrt{EG}\ \sin\alpha$$

and $$\lambda = \frac{1}{2H^2}\{2EF_1 - EE_2 - FE_1\}$$

$$= \frac{1}{2H^2}\left\{2E\left(\frac{\sqrt{G}}{2\sqrt{E}}E_1 + \frac{\sqrt{E}}{2\sqrt{G}}G_1\right)\cos\alpha - EE_2 - \sqrt{EG}\cos\alpha\, E_1\right\}$$

$$= \frac{1}{2EG\sin^2\alpha}\left\{\frac{E^{\frac{3}{2}}}{\sqrt{G}}G_1\cos\alpha - EE_2\right\}$$

Hence $k_{gu} = H\lambda E^{-\frac{3}{2}}$

$$= \frac{1}{2\sqrt{EG}\ \sin\alpha}\left\{\frac{G_1}{\sqrt{G}}\cos\alpha - \frac{E_2}{\sqrt{E}}\right\}$$

Similarly the geodesic k_{gv} of the family u=constant is given by

$$k_{gv} = \frac{1}{2\sqrt{EG}\ \sin\alpha}\left\{\frac{G_1}{\sqrt{G}} - \frac{E_2}{\sqrt{E}}\cos\alpha\right\}$$

$$= \frac{1}{\sin\alpha}\left\{\frac{G}{2G\sqrt{E}} - \cos\alpha\right\}$$

$$= \frac{1}{\sin\alpha}\{\gamma - \gamma'\cos\alpha\}, \text{ where}$$

γ and γ' stand respectively for $\dfrac{G_1}{2G\sqrt{E}}$ and $\dfrac{E_2}{2E\sqrt{G}}$.

According to the same notation

$$k_{gu}\{\gamma\cos\alpha - \gamma'\}\frac{1}{\sin\alpha}.$$

By the condition of the problem

$$\gamma' - \gamma \cos \alpha = P \sin \alpha$$

and

$$\gamma - \gamma' \cos \alpha = Q \sin \alpha$$

where P and Q are exclusively functions of u and v respectively

Hence
$$\gamma = \frac{Q + P \cos \alpha}{\sin \alpha} \text{ and } \gamma' = \frac{P + Q \cos \alpha}{\sin \alpha}$$

Now
$$2KH = \frac{\partial}{\partial u}\left\{\frac{F}{EH}E_2 - \frac{1}{H}G_1\right\} + \frac{\partial}{\partial \theta}\left\{\frac{2}{H}F_1 - \frac{1}{H}E_2 - \frac{F}{EH}E_1\right\}$$

$$= \frac{\partial}{\partial u}\left\{\frac{E_2}{E}\cot\alpha - \frac{G_1}{\sqrt{EG}\sin\alpha}\right\} + \frac{\partial}{\partial \theta}\left\{\frac{G_1}{G}\cos\alpha - \frac{E_2}{\sqrt{EG}}\right\}\frac{1}{\sin\alpha}$$

$$= \frac{\partial}{\partial u}\left\{\frac{E_2}{E}\cos\alpha - \frac{G_1}{\sqrt{EG}}\right\}\frac{1}{\sin\alpha} + \frac{\partial}{\partial v}\left\{\frac{G_1}{G}\cos\alpha - \frac{E_2}{\sqrt{EG}}\right\}\frac{1}{\sin\alpha}$$

$$= -\frac{\partial}{\partial u}(Q\sqrt{G}) - \frac{\partial}{\partial v}(P\sqrt{E})$$

$$= -\left\{Q\frac{G_1}{2\sqrt{G}} + \frac{PE_2}{2\sqrt{E}}\right\}$$

Therefore,
$$-2K = \left\{Q.\frac{G_1}{2G\sqrt{E}} + P.\frac{E_2}{2E\sqrt{G}}\right\}\frac{1}{\sin\alpha}$$

$$= \frac{1}{\sin^2\alpha}\{Q(P\cos\alpha + Q) + P(Q\cos\alpha + P)\}$$

$$= \frac{1}{\sin^2\alpha}\{P^2 + Q^2 + 2PQ\cos\alpha\}$$

And hence the theorem

Remarks on a Certain Lemma

BY

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(Lucknow University.)

In a previous paper published in this Bulletin (Vol XXVI, pp. 15 34), I gave two lemmas of which the second one is the following

If with each point of a set CG in (a, b) there is given one interval Δ with that point as left end point, and with each point of the complementary set G all intervals δ tending to zero (in length) with that point as left-end point, then, provided that G is a set of the first category in (a, b) there exists a chain of Δ and δ intervals reaching from a to b, and such that

$$m_e \Delta \geq (b-a)-\eta$$

where η is any arbitrary small positive number

It was pointed out (p 18) that the chain was not unique Moreover, numbers, of the second class had to be employed in the proof. Because of these two factors implicit faith could not be put on the above lemma and its corollaries, as my remarks (on the same page) would show I have now been able to construct a simple example which is in contradiction with the above lemma

Example.—Consider a non dense perfect set G of positive measure. To every point lying inside a contiguous interval of G let there be assigned the portion of the contiguous interval which lies to the right of that point Thus we have assigned to each point of CG one unique interval Δ with that point as left end point.

It is now easy to see that $\Sigma \Delta \leq$ the sum of the lengths of the contiguous intervals of G Thus

$$m_e \Delta \geq b-a-m$$

where m is the measure of the set G.

Remarks.—The above contradiction shows that we cannot make *unrestricted* use of numbers of the second class—especially the notion of one ordinal being greater than another as deduced from Cantor's first principle of generation

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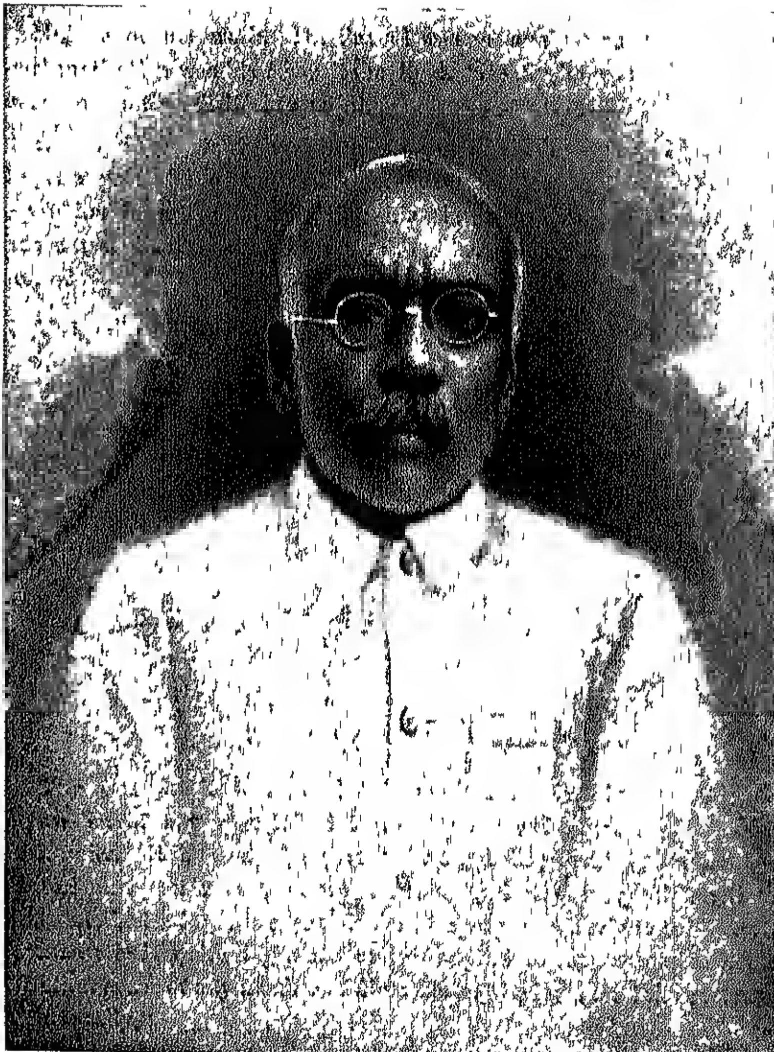

Dr GANESH PRASAD
President, Calcutta Mathematical Society,
1924-1935

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IN MEMORIAM

Dr. Ganesh Prasad

1876-1935

(President, Calcutta Mathematical Society, 1924-1935)

1. In the sudden death, on March 10, 1935, of Dr. Ganesh Prasad, Hardinge Professor of Higher Mathematics, Calcutta University has lost one of its most eminent teachers and the Calcutta Mathematical Society its distinguished leader. Born on the 15th November 1876 at Ballia, Agra Provinces, Dr. Prasad graduated with high honours from Allahabad University and after taking his M.A. from Calcutta and Allahabad Universities, and his Doctorate in Science at Allahabad University he proceeded to England with a Government of India stipend in 1899. He read at Cambridge with men like Hobson, Forsyth and Larmor and at Göttingen with Klein, Hilbert and Sommerfeld. After completing five years of study in Europe (1899-1904) he came back to his country of origin and took up the position of a lecturer at Queen's College, Benares. He had been temporarily a lecturer at Allahabad Kayastha Pathshala and Muir College before he left for England, and before he was appointed to the Hardinge Chair he was Ghose Professor of Applied Mathematics, Calcutta University and Principal, Benares Hindu University. It appeared that the profession of teaching was highly congenial to him, and though he took some part in the political life of his province for a brief spell, being elected to the legislative council of U. P., he gave the public life a go-by at the earliest opportunity and entered his homely study as the fit habitation of a scholar. A scholar's life he led up to the last moment when he collapsed while addressing an academic meeting at Agra.

A scholar's life is never rich in adventures except perhaps adventures of intellectual discovery. Dr. Ganesh Prasad has to his credit a full complement of important mathematical discoveries

2 Dr Ganesh Prasad's Contributions to Mathematics

Nature of his work. Dr. Prasad was an analyst pure and simple. Arithmetisation was his favourite method. Really he was Wierstrass's successor—although Klein's pupil

I First important paper in Messenger of Mathematics (1901), p. 8. On the potential of Ellipsoids of variable densities. Method of expansion in series adumbrating his later works on Summation theorems and asymptotic expansions Starting from Maclaurin expansion of

$$V(1)=\iiint F(f-x,\, g-y,\, h-z)\,dxdydz$$

$$=\sum_0^\infty \left[\iiint \frac{\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right)^{2i}}{2i} \,.\, F(f,\, g,\, h)\,dxdydz\right]$$

he derives Dyson's formula

$$V\left\{\phi\left(\frac{x}{y}, \frac{y}{b}, \frac{z}{c}\right)\right\}=2\pi abc\int_0^\infty \frac{P}{\sqrt{Q}}\,\theta_1\left\{\sqrt{(P\psi\delta)}\right\}\times$$

$$\left\{\phi\left(\frac{af}{a^2+\psi}, \frac{bg}{b^2+\psi}, \frac{ch}{c^2+\psi}\right)\right\}d\psi$$

The remarkable feature of this method is that it furnishes an expansion of any algebraic integral functions in series of spherical harmonics It is not necessary to knew any thing about the singularities of integrands because we are concerned with integral functions. The method of getting round the improper behaviour of certain parameters is quite ingenious and it applies to the expansion in any function space of any number of dimensions Cayley's mistake corrected incidentally.

II Next paper Constitution of matter and analytical theories of Heat This paper is now quoted as an authoritative solution of a difficult question in mathematical physics. Klein has pointed out that it is in this paper that a satisfactory solution of the physical problem has been given It is well known that starting from Fourier

down to Franz Neumann, Boussinesque, Poincaré, Heusl and Lamé every mathematician dealing with the problem of heat conduction has taken into consideration solutions of the differential equation concerned in a smoothed out uniformly convergent form. Dr Prasad has shown that if you start with integral expansion method then for ranges every where dense or non-dense you get a logically valid result So the real, discontinuous distribution of matter may be taken into account and the mathematical difficulties experienced by his predecessors are removed

III. Expansion of arbitrary functions in a series of spherical harmonics. 1912 (Math Ann) This is a very important result quoted in Hobson's recent book. The process is simplicity itself The process gives very important clues to some of the atomic phenomena Now it is common knowledge that in atomic physics we can observe the macroscopic results of energy transformation and can never penetrate into the microscopic phenomena whatever refined methods we may adopt The famous Heisenberg-uncertainty formula has set a limit to causality But if you are allowed to take an arbitrary average value you can expand it into particular infinite asymptotic series of a given type The type may easily be chosen to be that of spherical harmonics. Here we have an indication of an important result later discovered by Dr. Prasad regarding non orthogonal functions which says that $\int Q_m Q_n d\tau \neq 0$, $\int | Q_m |^2 d\tau \neq 1$, whatever m and n may be. This may serve to give an interpretation of the tailing off of band spectra of alkali metals.

IV. On the failure of Lebesgue's Criterion.

The smoothing out process recommended by Lebesgue was improved by Fejér But there was an oversight which was detected by Prasad and when communicated to Lebesgue himself, the latter owned his mistake. Dr Prasad has shown how the behaviour of functions with discontinuities of the 2nd kind should be controlled Later in 1933 the important paper on a connected topic was published, *viz*., on Lebesgue's integral mean-value for a function having a discontinuity of the 2nd kind.

V I shall close this account by mentioning an epoch-making paper published in Crelle's Journal in 1929 on the differentiability of the integral-function The logical acumen of the writer can only be appreciated by the leading pure mathematicians of our time. It

was in connexion with researches of this type that Pringsheim and Tonelli accepted Prasad as their peer

Perhaps the lasting contribution to mathematics would have been the mammoth paper, as I have called it in another place, if only the paper on expansion of an arbitrary function in infinite zeros had been finished But the fates have willed it otherwise

VI. List of works

(a) Original

Various papers in

Messenger of Mathematics
Mathematische Annalen
Rendiconti del circolo matematico di Palermo
Proceedings of Benares Mathematical Society
Bulletin of Calcutta Mathematical Society
Bulletin of American Mathematical Society
Crelle's Journal

On the function θ in the mean-value theorem of the Differential Calculus (Commemoration Volume of the Bulletin of the Calcutta Mathematical Society, 1929)

On the differentiability of the integral-function (Crelle's Journal, Vol 160, 1929)

On Rolle's function as multiple-valued function (Proceedings of the Benares Mathematical Society, Vol. X, 1929).

On the Zeros of Weierstrass's non differentiable function (Proc, B M, S, Vol XI, 1930)

On the nature of θ in the mean-value theorem of the Differential Calculus (Bulletin of the American Mathematical Society, Vol. XXXVI, 1930)

On the summation of infinite series of Legendre's functions, first paper (Bulletin C, M S, Vol. XXII, 1230)

On the determination of $f(h)$ corresponding to a given Rolle's function $\theta(h)$ when it is multiple valued (Proc., B. M. S., Vol XII, 1931).

On non-orthogonal systems of Legendre's functions (Proc., B. M. S. Vol. XII, 1931).

On the summation of infinite series of Legendre's functions, second paper (Bull. C M S, Vol XXIII, 1931)

On Rolle's function θ in the mean-value theorem for the case of a nowhere differentiable $f'(x)$ (Bull C M S., Vol XXIII, 1931)

On the differentiability of the indefinite integral and certain summability criteria (Address delivered in 1932 to the Mathematical and Physical Section of the Science Congress)

On Lebesgue's integral mean value for a function having a discontinuity of the second kind (Proc, B M S, Vol. XIV, 1933).

On Lebesgue's absolute integral mean value for a function having a discontinuity of the second kind (Special Memorial Volume of the Tohoku Mathematical Journal in honour of Prof Hayashi, 1933)

Hobson, Presidential address on the life and work of the late Prof Hobson (Bull C M S, Vol XXV, 1933)

(b) Didactic

Differential Calculus 1909

Integral Calculus 1910

An introduction to Elliptic Functions, &c 1928

Spherical Harmonics, &c, 2 parts 1930-32

Six lectures on recent researches about the mean-value theorem of the Differential Calculus

Six lectures on recent researches in the theories of Fourier Series 1928.

(c) Historical.

1. Mathematical Physics and Differential Equations at the beginning of the 20th century

2. Some great mathematicians of the 19th century (2 volumes published)

Dr. Prasad was a fascinating teacher On students who took his course he left a lasting impression as a master of his subject and inspired in them his own deep love for mathematics.*

S. C. BAGCHI.

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* The substance of this was delivered as a lecture at a memorial meeting held in the hall of the Indian Association for the Cultivation of Science on the 9th April, 1935.

CORRECTIONS

On the Product of Parabolic Cylinder Functions, by Dr S. C. Dhar, Vol. XXVI, pp. 57-64.

P. 59, 1st line from the top, please *read* Σ' *for* Σ

P. 61, 9th line from the top, please *read* $D_n(X)D_m(x)$

for $D_n(x)D_m(x)$.

5th line from the bottom, please *read*

$$e^{-\frac{1}{4}T^2-\frac{1}{4}U^2+\frac{1}{2}i(X+x)U+\frac{1}{2}i(X-x)T}dUdT$$

for

$$e^{-\frac{1}{4}T^2-\frac{1}{4}U^2+\frac{1}{2}i(X+x)U+\frac{1}{2}i(X-x)T}dUdT.$$

2nd line from the bottom, please *read* "Mitra,‡ *loc. cit.*"

for "Ditto Ditto"

P. 62, 4th line from the top, please *read* $D_n(X)\ D_m(x)$

for $D_n(X)D_m(x)$.

1st line from the bottom, please *read* $F(-r, -m\,;\, n-r+1\,;\, -1)$.

for $F(-r, -m\,;\, n-r+i\,;\, -1)$.

P. 63. 2nd line from the bottom, please *read* *Phil. Mag.*

for *Phil. Map.*

1st line from the bottom, please *read* " to be published soon."

for " to be published by Shastri."

KRISHNAKUMARI GANESH PRASAD PRIZE AND MEDAL

(for 1938)

The Council of the Calcutta Mathematical Society invites "Thesis" embodying the result of Original research or investigation in the following subject, for the Krishnakumari-Ganesh Prasad Prize and Gold Medal for the year 1938

Lives and works of the ten famous Hindu Mathematicians :—

(1) Aryabhatta, (2) Varahamihir, (3) Bhaskara I, (4) Lalla, (5) Brahmagupta, (6) Sridhar, (7) Mahabir, (8) Sripati, (9) Bhaskara II, (10) Narayana

The last day of submitting the thesis for the present award is 31st March, 1938 Three copies of the thesis (type written) are to be submitted

The competition is open to all nationals of the world without any distinction of race, caste or creed

All communications are to be sent to the Secretary, Calcutta Mathematical Society, 92, Upper Circular Road, Calcutta

14

On the rate of Disappearance of the Proper Motion of a Nebula according to the Expansion Theory

BY

N. R. Sen

1.

The statistical study of isolated nebulae and clusters of nebulae, based on the estimation of their distances from the very probable existence of an upper limit to the absolute luminosity of involved stars as well as of the nebulae themselves, shows a very good proportionality between the distances of the nebulae and their radial velocities. Recent observations have further confirmed this velocity distance relation.* The statistical examination has been extended, in addition to groups and clusters of nebulae to isolated nebulae, and the results are in good agreement with the above relation. The scatter round the mean value is ordinarily reasonably small, but there are cases, when deviations cannot probably be wholly accounted for by uncertainties of the determinations of distances. The residuals, in many such cases, according to Hubble and Humason, should represent the average peculiar motions of the individual nebulae and of the groups.† In some cases the deviations from the average value are abnormally large. For instance, the second group of the ten isolated nebulae studied by Hubble and Humason, which has an average velocity of +3420 km/sec for an average distance of 4·2 megaparsecs, shows a discrepancy of about 1000 km/sec.‡ There is no doubt that much of these deviations from the

* E. Hubble and M. Humason, The velocity distance relation for isolated extragalactic nebulae, Proc. Nat. Acad. Sc., **20**, p. 264.

† Proc. Nat. Acad. Sc., **15**, pp. 168-73.

‡ Astrophys. J., **74**, pp. 43-80.

average regularity represented by the velocity-distance relation is to be ascribed to the peculiar motions of the nebulae themselves.

In the expanding model of the Universe irregularities, such as a velocity relative to the mean motion of local matter (which is given fixed co-ordinates) is known to decrease gradually, so that all matter ultimately tends to come to rest in the co-ordinate system used, that is relative to the average nebula. It is, however, doubtful if the rate of this decrease has been properly appreciated. The mathematical treatment is beset with great difficulty on account of our ignorance of the function $R(t)$, the "curvature of space." An attempt has been made by extrapolation from existing data, firstly, to calculate certain limits within which this rate must lie, for instance an upper and a lower limit to the time in which a proper motion will be decreased by a certain percentage and the corresponding distances have been obtained. The assumption at the basis of the numerical calculations is that R/R has the uniform value which is calculable from existing observational data. Secondly, it is pointed out that assuming the Universe started on its present career of expansion in the finite past from some singular state, the appearance of a nebula with a definite irregularity at a definite distance can suggest an upper limit to the time scale. The study of the proper motions of the nebulae, then, in a certain sense, will give an indication of the maximum age of the expanding Universe. There, of course, remains an uncertainty regarding the present value of $R(t)$. But the result depends on the ratio $R(t)/R_0$ which, according to all methods of calculation is very probably a small figure. It is needless to emphasize that at the present stage, when speaking of the age of the expanding Universe we mean the order of the numerical figure, not the exact number. While other methods suggest an age near about 10^9 to 10^{10} years, the point of the present method is that in addition to retaining this order it suggests a figure as a probable maximum age.

2

We shall first work out the formulae for determining the minimum age. The metric field of the Universe is taken in the form *

* R. C. Tolman, Relativity, thermodynamics & cosmology, p. 370.

$$ds^2 = -\frac{e^{g(t)}}{[1+r^2/4R_0^2]^2}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+dt^2 \quad \dots \quad (1)$$

ere the "radius" $R(t)$ is given by

$$R(t)=R_0 e^{\frac{1}{2}g(t)} \quad . \quad (2)$$

the expanding type of the Universe, $g(t)$ is a monotone increasing ction of t The fourth equation for a geodesic in (1) gives on egration

$$\left(\frac{dt}{ds}\right)^2-1 = Ae^{-g(t)}$$

ere A is a constant of integration The interpretation of A has n given by Tolman † thus If a free particle has got a velocity u /sec relative to a local observer having the mean motion of matter his neighbourhood, then

$$Ae^{-g(t)} = \frac{u^2/c^2}{1-u^2/c^2} \quad . \quad (4)$$

m which it immediately follows that u will be decreasing with ne, $g(t)$ being monotone increasing For simplicity let us consider e case of radial motion We have

$$\left(\frac{ds}{dt}\right)^2 = 1-\frac{e^{g(t)}}{[1+r^2/4R_0^2]^2}\left(\frac{dr}{dt}\right)^2 \quad (5)$$

d on substitution of (3)

$$\pm\frac{1}{1+r^2/4R_0^2}\left(\frac{dr}{dt}\right) = \frac{\sqrt{A}\,e^{-g(t)}}{\sqrt{1+Ae^{-g(t)}}} \quad \dots \quad (5')$$

rom (2) and (4), since A is a positive quantity

$$\sqrt{A} = \frac{R(t)}{R_0}\frac{|u/c|}{\sqrt{1-u^2/c^2}} \quad \dots \quad (6)$$

† l.c., p 985

The substitution of this value and (4) in equation (5) gives ultimately

$$\frac{\pm\frac{dr}{dt}}{1+r^2/4R_0^2} = |u/c|\,\frac{R_0}{R(t)}, \quad \ldots \quad (7)$$

whence integrating we have

$$\pm 2R_0(\tan^{-1} r/2R_0 - \tan^{-1} r_0/2R_0) = \int_{t_0}^{t} |u/c|\,\frac{R_0}{R(t)}\,dt \quad . \quad (7a)$$

For distances of a few million parsecs to which observations are confined $r << R_0$, (7) and (7a) can be replaced by

$$\pm\left(\frac{dr}{dt}\right) = |u/c|\,\frac{R_0}{R(t)} \quad \ldots \quad (8)$$

and

$$\pm(r - r_0) = \int_{t_0}^{t} |u/c|\,\frac{R_0}{R(t)}\,dt. \quad \ldots \quad (8a)$$

Equation (8a) cannot be integrated further though it easily lends itself to further approximations. But we note that (8) and (8a) are not immediately applicable to observational data, since t_0 and t_1 are not the times of observation at the origin They are times by observer's clock when light left the nebula while occupying positions r_0 and r respectively The difference of these two times as perceived by the observer will not be the same as that given by (8a) when regarded as an equation for $(t - t_0)$ To get the observer's interval we modify the calculation thus Calling t' the time of arrival at the observer at the origin of light leaving the nebula at time t, we have

$$\pm dr = |u/c|\,\frac{R_0}{R(t)}\cdot\frac{dt}{dt'}dt'$$

The ratio (dt/dt') is of the form

$$\left(\frac{dt}{dt'}\right) = \frac{R(t)}{R(t')}\,\frac{1}{(1+u_r/c)},$$

where u_r is the radial component of the velocity of the nebula relative to local matter at rest in the co ordinate system. Substitution of this in the previous equation and integration gives

$$\pm\{r(t'_1)-r_0(t'_0)\} = \int_{t'_0}^{t'_1} \frac{|u/c|}{(1+u_r/c)} \frac{R_0}{R(t')}\, dt' \qquad \ldots \quad (9)$$

On the left hand side the notation is slightly altered, where it means the difference of the co-ordinate distances *observed* at times t'_1 and t'_0. The right hand side cannot be integrated further. But since $|u/c|$ is monotone decreasing and $R(t')$ is monotone increasing we have when the outward velocity of the nebula is greater than the velocity of expansion

$$(r'_1-r'_0) > \frac{|u/c|_{t'_1}}{1+\overline{u_r/c}} \frac{R_0}{R_1} (t'_1-t'_0)$$

where u_r/c has been replaced by a mean value $\overline{u_r/c}$ in the interval t'_0 and t'_1. From this an upper limit to the interval can be obtained as follows:

$$(t'_1-t'_0) < \frac{r'_1-r'_0}{|u/c|_{t'_1}} \{1+\overline{u_r/c}\} \frac{R(t'_1)}{R_0}$$

$$< \frac{r'_1}{|u/c|_{t'_1}} \{1+\overline{u_r/c}\} \frac{R(t'_1)}{R_0} \sim \frac{r'_1}{|u/c|_{t'_1}} \frac{R(t'_1)}{R_0} \qquad \ldots \quad (10)$$

The neglect of $\overline{u_r/c}$ is ordinarily justifiable as the velocity of proper motion is, according to the existing observational material, small compared to the velocity of light.

If we assume the expansion started at a definite epoch t_0, and u to be the proper velocity of a nebula at time t_1 occupying the co-ordinate position r_1, equation (10) gives an upper limit to the age of expansion in the experience of an observer at the origin. This equation in fact states, that *even if we assume the nebula to have been present at the origin at the time when the expansion started and to have been moving outward in our co-ordinate system since then, it could not have appeared at the co-ordinate position* r_1 *in a time (in the experience of the observer) greater than that given by the right-hand side of* (10).

In applying this formula to the existing observational material we have to note several points. First it has been shown by Tolman *

* *L.c.*, pp 462-66.

that the co-ordinate distance r and the astronomically measured distance d are related as follows:

$$\frac{r}{1+r^2/4R_0^2} = d \cdot \sqrt{\frac{\lambda}{\lambda+d\lambda}} \cdot$$

For distances of a few million parsecs $r << R_0$, so that the denominator on the left may be replaced by unity. Hence

$$d \sim r \sqrt{1+\frac{d\lambda}{\lambda}} \,.$$

Even up to 20 million parsecs $(d\lambda/\lambda)$ is less than 0·01. The correction to r will be less than 2 per cent, which far exceeds the degree of accuracy of observational data. It is thus reasonable to identify r with d for such small distances.

With this interpretation of r as distance in an Euclidean space, it is permissible to interpret the bracketed expression in (1) as square of the element of length $d\sigma$ in the same space. Though the Doppler shifts mentioned in the previous sections measure only the radial velocities, there is no reason to believe that the peculiar motions of the nebulae are all radial. Equation (3) is applicable to all cases but equation (5) in a general case can be modified by replacing dr by $d\sigma$, the element of path in the Euclidean space. The corresponding changes to be introduced in the formulae are $\sigma_1-\sigma_0$ in place of r_1-r_0 in (8a) and σ_1 in place of r'_1 in (10). For small lengths $\sigma_1-\sigma_0$ may be identified with the chord joining the two points of the path, which is certainly less than r_0+r_1 and so less than $2r_1$. Thus we again get to the same formula (10), only the upper limit on the right is doubled, to be on the safe side. The existence of a cross-radial component in $|u/c|$ in the denominator of (10) will not affect the upper limit when in the calculations u is replaced by the radial component.

Secondly, in calculating u we should remember that the calculation of the radial velocity is made just as if everything takes place in ordinary Euclidean space. The Doppler effect, as measured by the terrestrial observer, is converted by the ordinary method into a velocity. We shall for the moment consider that relative to the terrestrial observer, the mean motion of matter in the neighbourhood of the nebula is represented by the appropriate velocity of recession at that distance according to the velocity-distance relation and the measured velocity of the nebula is its velocity relative to the same

observer The difference of these two velocities will give u.* As these velocities are small compared to light velocity, the difference is taken in the ordinary manner

On the righthand side of (10), $R(t')/R_0$ is an uncertain factor of which no definite numerical value is known. But in all probability, as suggested by other methods of calculation the value of this ratio does not exceed a double-figured number near about ten and is even suspected to be only about 2 In the following calculations no numerical value has, however, been substituted for this ratio

The following table shows the results of calculation according to formula (10) The nebulae have been so selected that there is in every case a large discrepancy between the observed velocity corresponding to the red shift and the velocity calculated from the photographic magnitude and the velocity distance relation so that there is a large probability that this discrepancy is mainly due to the proper motion of the nebula rather than to observational errors and wrong estimation of distances. Here m_{pg}= photographic magnitude, d= distance in 10^6 parsecs, v= velocity in km/sec deduced from observed red shift, V= velocity calculated from velocity distance relation, namely, 560 km/sec per megaparsec, u= velocity of proper motion The last column tabulates the value of Θ given by (13) as

$$\Theta = \frac{t'_1}{|u/c|_{t'_1}} > \frac{t'_1 - t'_0}{R(t'_1)/R_0}$$

* Justification for this can be easily obtained thus: If t_1 be the time when light leaves a nebula and t_2 the time when it arrives on the earth, we have the relation (Tolman, R T C, p 390)

$$(\lambda+\delta\lambda)/\lambda = e^{\frac{1}{2}\{g(t_2)-g(t_1)\}}\left(1+\frac{u_r}{c}\right)(1-u^2/c^2)^{-\frac{1}{2}} \qquad \dots \quad \text{(A)}$$

In this $(\delta\lambda/\lambda)$ may be considered as the total red shift, being the sum of $(\delta\lambda/\lambda)_e$ due to cosmic expansion, and $(\delta\lambda/\lambda)_p$ due to the proper motion of the nebula Also

$$e^{\frac{1}{2}\{g(t_2)-g(t_1)\}} = \frac{R_2}{R_1} = 1+(\delta\lambda/\lambda)_e.$$

In (A) the last factor on the right may be replaced by unity We have then

$$u_r/c = \{1+(\delta\lambda/\lambda)_t\}\{1+(\delta\lambda/\lambda)_e\}^{-1}-1$$
$$\sim (\delta\lambda/\lambda)_t-(\delta\lambda/\lambda)_e$$

where the suffix t represents the total shift. The relative radial velocity is thus obtained by subtracting from the observed Doppler effect the Doppler effect of the general expansion according to the velocity-distance relation.

setting an upper limit to the age measured in terms of $R(t'_1)/R_0$.

N G C	m_{pg}	d in 10^6 parsecs	v in km/sec	V in km/sec	u in km/sec	Θ
1407	11 5	1 1	+ 2000	+ 615	+1385	0 8 10^9 years
157	11 2	1 0	+ 1800	+ 560	+1240	0 8 $\cdot10^9$ "
1084	11 2	1 0	+ 1450	+ 560	+ 890	1·1 $\cdot10^9$ "
5982	12 7	2 0	+ 2900	+1120	+1780	1·1 $\cdot10^9$ "
N_2	17·5	17·1	+19000	+9600	+9400	1 7 $\cdot10^9$ "
5005	10·6	0 76	+ 1030	+ 425	+ 605	0 76$\cdot10^9$ "
4151	10 9	0 87	+ 1050	+ 485	+ 565	1 5 $\cdot10^9$ "
4649	9 8	0·52	+ 1130	+ 406	+ 730	0 7 $\cdot10^9$ "
6703	13 6	3 0	+ 2280	+1680	+ 600	4 9 10^9 "
6661	14 0	3 6	+ 4170	+2020	+2150	1 6 10^9 "
6710	15·0	5 8	+ 5380	+3250	+2130	2 5 $\cdot10^9$ "

Considering the frequent occurrence of the value of Θ near about 1 10^9 we may take the upper limit to the age of the Universe to be given by about $\frac{R(t'_1)}{R_0}\cdot 10^9$ years (so far as can be judged from the present data)

3. *Motion for the simple model* $g(t)=2kt$.

It has been stated that our ignorance of the nature of the function $R(t)$ does not allow us to proceed further than (8). But to form an idea of the time and rate at which the proper motions tend to disappear we work out the case of the simplified model

$$g(t)=2kt \quad \ldots \quad (11)$$

for which the coefficient of expansion $\dot{R}/R$ is equal to a constant k whose value calculated from present day observations is

$$5{\cdot}71\times10^{-10}(yrs)^{-1}.$$

Equation (5′) can be integrated, remembering (11) in the form

$$\int_{r_0}^{r} \pm \frac{dr}{1+r^2/4R_0^2} = -\frac{1}{2k\sqrt{A}} \int_{t_0}^{t} \frac{-gAe^{-g(t)}}{\sqrt{1+Ae^{-g(t)}}}\, dt$$

Integrating and substituting from (4), when $r << R_0$,

$$\pm(r-r_0) = -\frac{1}{R\sqrt{A}}\left[\frac{1}{\sqrt{1-u^2/c^2}} - \frac{1}{\sqrt{1-u_0^2/c^2}}\right]$$

which on further substitution from (6) leads to

$$\pm(r_1-r_0) = \frac{1}{R}\,\frac{1}{|u/c|}\left[\sqrt{\frac{1-u^2/c^2}{1-u_0^2/c^2}}-1\right]\frac{R_0}{R(t)} \qquad \ldots \quad (12)$$

Usually u/c and u_0/c are both small, in such cases expressing r_1-r_0 *in parsecs* we have

$$\pm(r_1-r_0) = 2{\cdot}7\times10^8\ \frac{1}{|u/c|}\,(u_0^2/c^2-u^2/c^2)\,\frac{R_0}{R(t)}. \qquad \ldots \quad (13)$$

This formula enables us to calculate in terms of the ratio $R_0/R(t)$, the co-ordinate distance r_1-r_0 traversed by a nebula as its velocity decreases from u_0 to u. It is necessary to point out that this co-ordinate "distance" is really an interval whose end-points are occupied by the nebula at *two different epochs*, so that r_1-r_0 is not the actual distance covered by the nebula. We may imagine two hypothetical nebulae having no proper motion at the points r_0 and r_1. The distance r_1-r_0 in parsecs in (13) is the distance between those two nebulae, say at the first epoch. The formula (13) shows that the moving nebula has just overtaken the hypothetical nebula at r_1 when the proper motion of the moving nebula has fallen from u_0 to u. The formula involves the rather inconvenient factor $R_0/R(t)$, and gives a result only in terms of this ratio.

Formula (6) can be used for estimating the time in which the proper velocity of a nebula is reduced by a certain per cent., since on the right-hand side A is a constant for the motion of a nebula. But the interval in this case is given in terms of the radii of the Universe at the two epochs.

For the particular model $g(t)=2kt$ we can form some idea of the time interval in which the decrease in proper velocity takes place by

integrating 7(*a*). If $(r_0, r) << R_0$ we have from (7*a*)

$$\pm(r-r_0) = \int_{t_0}^{t} \left|\frac{u}{c}\right| e^{-kt} dt$$

$$= \frac{|u'|}{ck} \left(e^{-kt_0} - e^{-kt}\right) \qquad \ldots \quad (15)$$

where $|u| < |u'| < |u_0|$. Combining this equation with (12) which for $|u/c| << 1$ can be written as

$$\pm(r-r_0) = \frac{1}{2kc} \frac{e^{-kt}}{|u|} (u_0^2 - u^2)$$

we have

$$e^{k(t-t_0)} = 1 + \frac{u_0^2 - u^2}{2|uu'|}$$

whence

$$t - t_0 = \frac{1}{k} \log\left(1 + \frac{u_0^2 - u^2}{2|uu'|}\right), \quad |u| < |u'| < |u_0| . \quad \ldots \quad (16)$$

This time interval is not that of the observer who has to take along with it a Doppler effect factor. But for our purpose of a rough estimate of the time interval this is enough.

The random proper motions of the nebulae have a tendency to be gradually regularised. The extreme slowness of the process of regularisation is shown by (16). For instance, if we take $|u_0| = 100$ km/sec, and $|u| = 99$ km/sec, equation (16) shows that this diminution of 1% of the velocity takes place in time $(t-t_0) \sim 0{\cdot}2 \times 10^8$, years. A diminution of 10% of the velocity will be brought about in time $(t-t_0)$ years where

$$0{\cdot}92 \times 10^9 \text{ years} < t - t_0 < 1{\cdot}6 \times 10^9 \text{ years.}$$

Applying formula (13) we find that in the first case while 1% of the velocity disappears, the nebula traverses a co-ordinate distance $r - r_0 = 1808\ R_0/R(t)$ parsecs, and in the latter case, for a diminution of the 10% of the velocity, $r - r_0 = 19{,}000\ R_0/R(t)$ parsecs nearly.

These figures only point to the extreme slowness of the process of regularisation mentioned above and show the minute deviation from the classical law of inertia for local observers in the new scheme of the expansion theory.

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15

A NOTE ON THE AREA CENTROID OF A CLOSED CONVEX OVAL

BY

R. C. BOSE AND S. N. ROY

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Introduction.—Three different kinds of centroid are connected with a closed convex oval curve V. The centroid of the perimeter when to each point we associate a density equal to the curvature at the point, is known as the *curvature centroid** of the oval. The centroid of the perimeter when to each point we associate a uniform density may be called the *perimeter centroid*. The centroid of the area of the oval, on the assumption of uniform density, may be called the *area centroid*. When we consider a system of curves parallel to V, the curvature centroid remains invariant, while according to a theorem of Kubota† the locus of the perimeter centroid is a straight line. The object of the present note is to study the corresponding locus for the area centroid. We show that the locus in question is a conic, the complete relation between the curvature centroid, and the loci of the other two centroids being given by the following theorem:—

For a system of curves parallel to a convex oval, the curvature centroid is a fixed point G_0, the locus of the perimeter centroid is a straight line Σ_1, passing through G_0, and the locus of the area centroid is a conic Σ_2, touching Σ_1 at G_0. If G_1 be the perimeter centroid and G_2 the area centroid of the same curve of the system, then the tangent to Σ_2 at G_2 passes through G_1.

Corresponding to Kubota's theorem, that if for any one oval of the system, the perimeter centroid coincides with the curvature centroid

* J. Steiner: Von dem Krümmungsschwerpunkte ebener Kurven. Crelle J. 21 (1838).

† T. Kubota: Über die Schwerpunkte der convexen geschlossenen Kurven und Flächen. Tohoku Math. J. 14, (1918), 20-27.

then the same is true for every oval of the system, we now get: (1) *if for any oval of the system the three centroids lie on a line, then they lie on this line for every oval of the system,* (2) *if for any oval of the system the three centroids coincide, they coincide for every oval of the system*

Corresponding to our theorem* that for a system of parallel convex ovals, the distance between the perimeter and the curvature centroid varies inversely as the perimeter, we now get *for a system of parallel convex ovals the area of the triangle formed by the three centroids varies inversely as the product of the perimeter and the area*

If ρ_1 and ρ_2 be the maximum and the minimum radii of curvature of the closed convex oval V, then the parallel curve at a distance h, (h being measured positively along the outward normal) will be an oval only when h does not lie between $-\rho_1$ and $-\rho_2$. Let Σ'_2 be that arc of the conic Σ_2 (which is the locus of the area centroids of curves parallel to V), which corresponds to values of h not lying between $-\rho_1$ and $-\rho_2$ † Both the curvature centroid G_0 and the area centroid G_2 of V lie on Σ'_2. We prove that, *if G be any point on Σ'_2,* (1) *at least four normals can be drawn from G to V,* (2) *there exist at least three pairs of parallel tangents to V, such that the tangents belonging to the same pair are equidistant from G,* (3) *at least three chords of V are bisected at G*. The properties (1) and (2) were proved for the curvature centroid G_0 by Hayashi,‡ while one of the authors proved the property (3) for G_0 and the properties (2) and (3) for G_2, in a recent note published in this bulletin §

I

Consider a closed convex oval V. Let (x, y) denote the co-ordinates of any point on V, and let p denote the length of the perpendicular form the origin on the positive tangent at (x, y), reckoned positively when the origin lies to the left of the tangent,

* R. C. Bose and S. N. Roy: Some properties of the convex oval with reference to its perimeter centroid, Bulletin Cal Math Soc., 27, *79-89* (1935)

† For values of h lying between $-\rho_1$ and $-\rho_2$, the curves parallel to V will be self cutting. The points of Σ_2 corresponding to these values of h can be still regarded as the area centroids of these curves, with suitable conventions as to the sign of the area.

‡ T. Hayashi: Some geometrical applications of Fourier series. Rend. Circ. Matem. Palermo. L (1926) *96-102*

§ R. C. Bose: A note on the convex oval, Bulletin Cal Math Soc., 27, *55-60* (1935).

Let ψ be the angle which the positive tangent makes with the positive direction of the x-axis. Then

$$p = x \sin\psi - y\cos\psi \qquad (1)$$

$$p' = x\cos\psi + y\sin\psi \qquad \ldots \quad (2)$$

Therefore

$$x = p\sin\psi + p'\cos\psi \qquad (3)$$

$$y = -p\cos\psi + p'\sin\psi \qquad (4)$$

where dashes denote differentiation with respect to ψ.

Let L denote the perimeter and A denote the area of V. If (x_0, y_0), (x_1, y_1) be the co-ordinates of the curvature centroid and the perimeter centroid respectively, we know that *

$$x_0 = \frac{1}{\pi}\int_0^{2\pi} p\sin\psi d\psi \qquad \ldots \quad (5)$$

$$y_0 = -\frac{1}{\pi}\int_0^{2\pi} p\cos\psi d\psi \qquad \ldots \quad (6)$$

$$x_1 = \frac{1}{L}\int_0^{2\pi} (p^2 - \tfrac{1}{2}p'^2)\sin\psi d\psi \qquad \ldots \quad (7)$$

$$y_1 = -\frac{1}{L}\int_0^{2\pi} (p^2 - \tfrac{1}{2}p'^2)\cos\psi d\psi \qquad \ldots \quad (8)$$

Let (x_2, y_2) be the co-ordinates of the area centroid of V.

Now
$$3Ax_2 = \int_0^{L} xp\,ds$$

$$= \int_0^{2\pi} xp\,\rho\,d\psi$$

$$= \int_0^{2\pi} (p^2\sin\psi + pp'\cos\psi)(p+p'')d\psi, \text{ from (1)} \quad \ldots \quad (9)$$

* T. Kubota, *Loc. cit.*; R. C. Bose and S. N. Roy, *Loc. cit.*

Now integrating by parts and noticing that the part outside the integral sign vanishes in each integration, when taken between the limits, we have

$$\int_0^{2\pi} p^2 p'' \sin\psi d\psi = -\int_0^{2\pi} p'(p^2 \cos\psi + 2pp' \sin\psi) d\psi$$

$$= -\tfrac{1}{3}\int_0^{2\pi} p^3 \sin\psi d\psi - 2\int_0^{2\pi} pp'^2 \sin\psi d\psi \qquad (10)$$

$$\int_0^{2\pi} p^2 p' \cos\psi d\psi = \tfrac{1}{3}\int_0^{2\pi} p^3 \sin\psi d\psi \qquad . \quad (11)$$

$$\int_0^{2\pi} pp'p'' \cos\psi d\psi = -\tfrac{1}{2}\int_0^{2\pi} p'^2(-p\sin\psi + p'\cos\psi) d\psi$$

$$= \tfrac{1}{2}\int_0^{2\pi} pp'^2 \sin\psi d\psi + \tfrac{3}{2}\int_0^{2\pi} p'^2 p'' \sin\psi d\psi. \qquad \ldots \quad (12)$$

Substituting from (10), (11) and (12) in (9) we have

$$x_2 = \frac{1}{3A}\int_0^{2\pi} \{p^3 - \tfrac{3}{2}p'^2(p - p'')\} \sin\psi d\psi \qquad .. \quad (13)$$

In the same way we have

$$y_2 = -\frac{1}{3A}\int_0^{2\pi} \{p^3 - \tfrac{3}{2}p'^2(p - p'')\} \cos\psi d\psi \qquad .. \quad (14)$$

II

Let $A(h)$ denote the area, and $L(h)$ the perimeter of the oval V_h parallel to V, at a distance h, h being reckoned positively along the outward normal. Let $x_2(h)$, $y_2(h)$ be the area centroid and $x_1(h)$, $y_1(h)$ the perimeter centroid of V_h. Then

$$A(h) = A + Lh + \pi h^2 \,;\; L(h) = L + 2\pi h \qquad \ldots \quad (15)$$

Hence from (13) we have

$$(A + Lh + \pi h^2) x_2(h) = \tfrac{1}{3}\int_0^{2\pi} \{(p+h)^3 - \tfrac{3}{2}p'^2(p + h - p'')\} \sin\psi d\psi$$

$$= Ax_2 + hLx_1 + \pi h^2 x_0,$$

from (13), (7) and (5).

Therefore $$x_2(h) = \frac{Ax_2 + hLx_1 + \pi h^2 x_0}{A + Lh + \pi h^2}, \qquad \dots \quad (16)$$

Similarly $$y_2(h) = \frac{Ay_2 + hLy_1 + \pi h^2 y_0}{A + Lh + \pi h^2} \qquad \dots \quad (17)$$

In the same way

$$x_1(h) = \frac{Lx_1 + 2\pi h x_0}{L + 2\pi h}, \qquad \dots \quad (18)$$

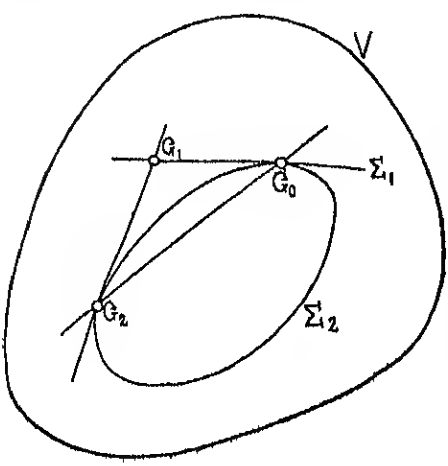
$$y_1(h) = \frac{Ly_1 + 2\pi h y_0}{L + 2\pi h} \qquad \dots \quad (19)$$

From (18) and (19) it is evident that the locus of $x_1(h)$, $y_1(h)$ is the straight line Σ_1 joining (x_0, y_0) and (x_1, y_1).

Eliminating h from (16) and (17) and replacing $x_2(h)$, $y_2(h)$ by x, y we find that the locus of the area centriod is the conic

$$\pi A \begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = L^2 \begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \qquad \dots \quad (20)$$

Let us denote this conic by Σ_2.



Denoting by G_0, G_1, G_2 the points (x_0, y_0), (x_1, y_1) and (x_2, y_2) respectively the form of the equation (20) at once shows that the conic Σ_2 touches the lines G_0G_1 and G_1G_2 at the points where they are cut by the line G_0G_2, *i.e.*, at the points G_0 and G_2 respectively. (See Figure.) We can thus state the following theorem :—

THEOREM.—*For a system of curves parallel to a convex oval, the curvature centroid is a fixed point* G_0, *the locus of the perimeter centroid is a straight line* Σ_1 *passing through* G_0 *and the locus of the area centroid is a conic* Σ_2 *touching* Σ_1 *at* G_0. *If* G_1 *be the perimeter centroid and* G_2 *the area centroid of the same curve of the system, then the tangent to* Σ_2 *at* G_2 *passes through* G_1.

III

Adhering to our previous notation, and further denoting by $\Delta(h)$ the area of the triangle formed by the three centroids of V_h, and by Δ the area of the triangle formed by the three centroids of V, we have from (16), (17), (18) and (19)

$$\Delta(h) = \frac{1}{2}\begin{vmatrix} \dfrac{Ax_2+hLx_1+\pi h^2x_0}{A+Lh+\pi h^2}, & \dfrac{Lx_1+2\pi hx_0}{L+2\pi h}, & x_0 \\[2ex] \dfrac{Ay_2+hLy_1+\pi h^2y_0}{A+Lh+\pi h^2}, & \dfrac{Ly_1+2\pi hy_0}{L+2\pi h}, & y_0 \\[2ex] 1, & 1 & 1 \end{vmatrix}$$

$$= \frac{AL}{2(L+2\pi h)(A+Lh+2\pi h^2)}\begin{vmatrix} x_2, & x_1, & x_0 \\ y_2, & y_1, & y_0 \\ 1, & 1, & 1 \end{vmatrix}$$

$$= \frac{\Delta AL}{A(h)L(h)}$$

Hence
$$\frac{\Delta(h)}{\Delta} = \frac{AL}{A(h)L(h)}. \qquad \ldots \quad (21)$$

We thus got the theorem :—

For a system of curves parallel to a convex oval, the area of the triangle formed by the three centroids varies inversely as the product of the perimeter and area

Suppose that as a special case the three centroids of V coincide, *i.e.*, $x_2=x_1=x_0$, $y_2=y_1=y_0$ Then the formulae (16)—(19) show that

$$x_2(h)=x_1(h)=x_0(h),\ y_2(h)=y_1(h)=y_0(h)$$

Hence, *if the three centroids of the convex oval* V *coincide, the same is the case for the three centroids of all curves parallel to* V.

Next suppose that as a special case, the three centroids of V lie on a line (this includes the case when any two centroids of V coincide). Let us choose this line as the axis of x. Then $y_2=y_1=y_0=0$. From (17) and (19) $y_2(h)=y_1(h)=0$ Hence if three centroids of the convex oval V lie on a straight line, the three centroids of any curve parallel to V lie on the same straight line

IV.

Hayashi * has shown that Blaschke's mechanical proof of the four cyclic point theorem is tantamount to the following :—

If a function $f(\phi)$ be one valued, continuous and periodic, with period 2π, and satisfy the conditions

$$\int_0^{2\pi} f(\phi) \frac{\cos}{\sin} \phi d\phi = 0 \qquad \dots \quad (22)$$

then the function has at least two maxima and two minima in the interval $(0, 2\pi)$.

Now if $r=F(\theta)$ be the polar equation of the closed convex oval V, with its area centroid G_2, chosen as the origin we have

$$\int_0^{2\pi} r^3 \frac{\cos}{\sin} \theta = 0. \qquad \dots \quad (23)$$

Hence r^3, and consequently r has at least two maxima and two minima in the interval $(0, 2\pi)$ Hence from the area centroid G_2, at least four normals can be drawn to the oval V.

Now consider a curve V_h parallel to V at a distance h from it If ρ_1 and ρ_2 be the maximum and minimum radii of curvature of V, then V_h will be an oval so long as h does not lie between $-\rho_1$ and $-\rho_2$. Let Σ'_2 be that part of the conic Σ which corresponds to values of h not lying between $-\rho_1$ and $-\rho_2$. As the curvature centroid G_0 corresponds to the value $h=\infty$ and the area centroid G_2 of V, corresponds to the value $h=0$, both G_2 and G_0 lie on Σ'_2 Now from any point G of Σ'_2 which corresponds to a given value of h, four normals can be drawn to V_h. Since these normals are also normals to V, we see that from any point of Σ'_2, at least four normals can be drawn to V. Again according to a theorem proved by one of the authors,† we

* T. Hayashi : *Loc. cit*

† R. C. Bose : *Loc. cit.*

can get three pairs of parallel tangents to V_h such that the tangents of each pair are equidistant from G The corresponding tangents to V also form pairs of parallel tangents, equidistant from G But the number of pairs of parallel tangents equidistant from G, is also equal to the number of chords bisected at G, according to another theorem in the paper referred to just before Hence three chords of V are bisected at G. Summing up we have the following result,—

If V is a closed convex oval, ρ_1 being the maximum and ρ_2 the minimum radius of curvature of V, and if h does not lie between $-\rho_1$ and $-\rho_2$, then the curves parallel to V at a distance h (h being reckoned positively along the outward normal) are convex ovals The locus of the area centroids of the system of parallel ovals is a part of a conic The curvature centroid G_0 and the area centroid G_2 of V are points on this locus Calling this locus Σ'_2, the oval V possesses the following properties in relation to any point G of Σ'_2

(a) At least four normals can be drawn from G to V

(b) There exist at least three pairs of parallel tangents to V such that the tangents belonging to each pair are equidistant from G

(c) At least three chords of V are bisected at G

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On the four centroids of a closed convex surface

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I

Introduction.

1. For a closed convex curve three different kinds of centroids exist;—(1) *The curvature centroid,* first investigated by Steiner,* (2) *the perimeter centroid,* (3) *the area centroid* Many interesting properties of the convex curve, with reference to the various centroids are known †

In the present paper we attempt to investigate the corresponding properties of the centroids of a closed convex surface (supposed to be regular analytic). A beginning in this direction has already been made by Hayashi ‡ and by Kubota, who showed that *the locus of the surface centroid of a system of surfaces parallel to a closed convex surface, is a conic section.*

* J Steiner: Von dem Krümmungsschwerpunkte ebener Kurven Crelle J. 21 (1838).

† T Kubota: Über die Schwerpunkte der convexen geschlossenen Kurven und Flächen. Tohoku Math J. 14, (1918), 20-27.

Meissner: Über die Anwendung von Fourier—Reihen auf einige Aufgaben der Geometrie und Kinematik, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich 54, (1909)

F. Schilling. Die Theorie u. Konstruktion der Kurve konstanter Breite, Zeitschrift für Math u. Physik (1914)

T Hayashi: On Steiner's curvature centroid. Science Reports of the Tohoku Imperial University, 13 (1924), 109-132.

R. C Bose and S. N. Roy: Some properties of the convex oval with reference to its perimeter centroid. Bull Cal. Math Soc., 27, (1935) 79-86.

R. C. Bose and S. N. Roy: A note on the area centroid of a closed convex oval. Bulletin Cal. Math Soc, 27, (1935) 111-118.

‡ T. Hayashi: *Loc. cit.*; T. Kubota: *Loc. cit.*

Corresponding to Steiner's curvature centroid, we can define two different kinds of centroids for the convex surface.* If to every point of the surface we associate a density equal to the Gaussian curvature $1/RR'$, where R, R' are the two principal radii of curvature at the point, then the centroid of the surface so weighted, we call *the Gaussian curvature centroid*. Again if to every point of the surface we associate a density equal to the mean curvature $\frac{1}{2}(1/R+1/R')$, then the centroid of the surface so weighted, we call *the mean curvature centroid* of the surface. The centroid of the surface when to every point we associate a uniform density, may be called the *surface centroid*. Similarly the centroid of the enclosed volume (a uniform density being supposed to be associated with each point), may be called the *volume centroid*.

2. Consider a regular analytic closed convex surface Ω. Let H denote the length of the perpendicular† from the origin, on the tangent plane at any point P of Ω. We can establish a (1, 1) correspondence between the surface, and the unit sphere, with the centre at the origin, in the sense that corresponding to the point P of the surface, we take the point P' on the unit sphere, P' being the point, where the half-line through the origin parallel to the outward normal to the surface at P, meets the unit sphere. Let ψ and ϕ, be the latitude and longitude on the unit sphere, where the section by the xy-plane is the equator and the section by the xz-plane is the meridian. To the point P' on the unit sphere we now associate the scalar quantity H, already defined. Any function of position on the unit sphere, and in particular H, can be looked upon either as a function of ψ and ϕ, or of α, β, γ, the direction cosines of the line joining the point to the centre. H may be called the *tangential function* (*stutzfunktion*) of the surface. Grad H would mean a vector lying in the tangent plane to the unit sphere at P', with cross meridianal and meridianal components

$$\frac{1}{\cos\psi}\frac{\partial H}{\partial \phi}, \frac{\partial H}{\partial \psi} \tag{1.11}$$

It is easily seen that grad H can also be regarded as a space vector with x, y, z components

* T. Bonnesen and W. Fenchel, Theorie der Konvexen Korper, 53

† If the origin be supposed to lie to the interior of the surface, then H is always positive. It is however possible to take the origin outside the surface, if suitable conventions of signs are adopted.

$$\left.\begin{aligned} &-\frac{\partial H}{\partial \psi} \sin\psi \cos\phi - \frac{\partial H}{\partial \phi}\,\frac{\sin\phi}{\cos\psi} \\ &-\frac{\partial H}{\partial \psi} \sin\psi \sin\phi + \frac{\partial H}{\partial \phi}\,\frac{\cos\phi}{\cos\psi} \\ &\frac{\partial H}{\partial \psi} \cos\psi \end{aligned}\right\} \quad \ldots \quad (1.12)$$

We may denote (as is usual) by $d\omega$ the element of surface on the unit sphere, so that

$$d\omega = \cos\psi\, d\psi\, d\phi. \qquad \ldots \quad (1.2)$$

3 If now r^0, r', r'', r''', *are the vectors from the origin to the Gaussian curvature centroid, mean curvature centroid, surface centroid and volume centroid, respectively*, we show that

$$4\pi r^0 = 3\int H n\, d\omega \qquad \ldots \quad (1.3)$$

$$2M_0 r' = \int (3H^2 - \nabla H) n\, d\omega \qquad \ldots \quad (1.4)$$

$$S_0 r'' = \int \{H^3 - H\nabla H + \tfrac{1}{4}\,(\text{grad } H.\ \text{grad})\nabla H\} n\, d\omega \qquad \ldots \quad (1.5)$$

$$\begin{aligned} 4V_0 r''' = &\int \{H^4 - 2H^2 \nabla H \\ &+ H\,(\text{grad } H.\ \text{grad})\ \nabla H + \tfrac{1}{2}(\nabla H)^2\} n\, d\omega \\ &+ \tfrac{1}{2}\int \{(\text{grad } H.\ \text{grad})\nabla H\}\ \text{grad } H\, d\omega. \end{aligned} \qquad \ldots \quad (1.6)$$

Here the integrations are over the whole unit sphere, n is the unit vector normal to the unit sphere, and ∇ is *Beltrami's first operator* given by

$$\begin{aligned} \nabla H &= \text{grad } H.\ \text{grad } H \\ &= \left(\frac{\partial H}{\partial \psi}\right)^2 + \frac{1}{\cos^2\psi}\left(\frac{\partial H}{\partial \phi}\right)^2 \end{aligned} \qquad \ldots \quad (1.7)$$

the dot signifying as usual the scalar product.

Again M_0 denotes *the integral of mean curvature* taken over the whole surface, S_0 denotes the surface area, and V_0 the volume enclosed by the surface. It is known that

$$M_0 = \int H \, d\omega \qquad \ldots \quad (1.81)$$

$$S_0 = \int (H^2 - \tfrac{1}{2} \nabla H) d\omega \qquad \ldots \quad (1.82)$$

and we show that

$$V_0 = \tfrac{1}{3} \int \{H^3 - \tfrac{3}{2} H \nabla H + \tfrac{1}{2} (\text{grad } H . \text{grad}) \nabla H)\} d\omega \qquad \ldots \quad (1.83)$$

Substituting from the above in (1.3), (1.4), (1.5) and (1.6), we get formulae *expressing* r^0, r', r'', r''' *purely in terms of the tangential function (Stutzfunktion)* H.

The proof of the results (1.3)—(1.6) and (1.83) mainly depends upon the following interesting result which may prove to be of wider application, than the use made of it in the present paper.

If U is a function of position on the unit sphere, being homogeneous in α, β, γ and of the nth degree in α, β, γ, then

$$\left.\begin{aligned} \int \frac{\partial U}{\partial \alpha} d\omega &= (n+2) \int \alpha U d\omega \\ \int \frac{\partial U}{\partial \beta} d\omega &= (n+2) \int \beta U d\omega \\ \int \frac{\partial U}{\partial \gamma} d\omega &= (n+2) \int \gamma U d\omega \end{aligned}\right\} \qquad \ldots \quad (1.9)$$

where the integrations are taken over the whole unit sphere.

4. We next go on to study the geometrical properties of the centroids. Corresponding to Meissner's theorem,* that the curvature centroid, and the perimeter centroid of a convex curve of constant breadth coincide, we prove that *the Gaussian and mean curvature centroids*

* Meissner *Loc. cit.*

of a convex surface of constant breadth coincide. Again we show that *the Gaussian curvature centroid of a system of parallel convex surfaces is a fixed point* G^0, *and the locus of the mean curvature centroid is a straight line* Σ'. Kubota* has already shewn that *the locus of the surface centroid is a conic* Σ''. We shew that *this conic touches* Σ' at G^0. Again we find that *the locus of the volume centroid is a rational space cubic* Σ''', *osculating* Σ'' at G^0. Finally *if* G^0, G', G'', G''' *be the four centroids for any one of the system of parallel surfaces*, we prove that *the tangent at* G'' *to* Σ'' *passes through* G', *the tangent at* G''' *to* Σ''' *passes through* G'' *while the osculating plane to* Σ''' *at* G''', *coincides with the plane* $G'G''G'''$. (See figure on page 145.)

II.

Operational Calculus

1. Let α, β, γ be the direction cosines of the normal at any point P of the surface, then α, β, γ are also the direction cosines of the line OP′, where O is the origin and P′ is the point on the unit sphere corresponding to P. Hence

$$\alpha = \cos\phi \cos\psi,\ \beta = \sin\phi \sin\psi,\ \gamma = \sin\psi, \qquad \ldots \quad (2.11)$$

where of course

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \qquad \ldots \quad (2.12)$$

It has already been noted that the tangential function (stutzfunktion) H can either be regarded as a function of ψ and ϕ, or of α, β, γ. Through the help of (2.12) we can make H, a homogeneous linear function of α, β, γ. Throughout this paper we shall always consider this to have been done. Then the following formulae (2.21)—(2.37) are known to hold.†

$$\left.\begin{aligned} x &= \frac{\partial H}{\partial \alpha} = H_1, \\ y &= \frac{\partial H}{\partial \beta} = H_2, \\ z &= \frac{\partial H}{\partial \gamma} = H_3 \end{aligned}\right\} \qquad \ldots \quad (2.21)$$

* Kubota. *Loc. cit.*

† W. Blaschke. Kreis und Kugel, *138 141.*

where partial differentiation with respect to α, β, γ is denoted by the suffixes 1, 2, 3. We shall use this notation throughout this paper. Also

$$\left.\begin{aligned} \alpha H_1 + \beta H_2 + \gamma H_3 &= H, \\ \alpha H_{11} + \beta H_{12} + \gamma H_{13} &= 0, \\ \alpha H_{21} + \beta H_{22} + \gamma H_{23} &= 0, \\ \alpha H_{31} + \beta H_{32} + \gamma H_{33} &= 0. \end{aligned}\right\} \qquad \ldots \quad (2.22)$$

If R, R′ be the principal radii of curvature of our surface, at any point, then

$$R + R' = H_{11} + H_{22} + H_{33} \qquad \ldots \quad (2.31)$$

$$= 2H + \Delta_2 H, \qquad \ldots \quad (2.32)$$

where Δ_2 is the well known Beltramian second operator given by

$$\Delta_2 = \frac{\partial^2}{\partial \psi^2} + \frac{1}{\cos^2\psi}\,\frac{\partial^2}{\partial \phi^2} - \tan\psi\,\frac{\partial}{\partial \psi} \qquad \ldots \quad (2.33)$$

$$= \text{div grad (in vector language)}. \qquad \ldots \quad (2.34)$$

$$RR' = \frac{H_{22}H_{33} - H_{23}^2}{\alpha^2} \qquad \ldots \quad (2.35)$$

$$= \frac{H_{33}H_{11} - H_{31}^2}{\beta^2} \qquad \ldots \quad (2.36)$$

$$= \frac{H_{11}H_{22} - H_{12}^2}{\gamma^2}. \qquad \ldots \quad (2.37)$$

2. Let U be any homogeneous function of α, β, γ of nth degree, where n is zero or is a positive or negative integer. Then

$$\frac{\partial U}{\partial \phi} = \frac{\partial U}{\partial \alpha}\,\frac{\partial \alpha}{\partial \phi} + \frac{\partial U}{\partial \beta}\,\frac{\partial \beta}{\partial \phi} + \frac{\partial U}{\partial \gamma}\,\frac{\partial \gamma}{\partial \phi}, \qquad \ldots \quad (2.41)$$

$$\frac{\partial U}{\partial \psi} = \frac{\partial U}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial \psi} + \frac{\partial U}{\partial \beta}\cdot\frac{\partial \beta}{\partial \psi} + \frac{\partial U}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial \psi}, \qquad \ldots \quad (2.42)$$

or from (2 11)

$$\frac{\partial U}{\partial \phi} = -\frac{\partial U}{\partial \alpha}\sin\phi\cos\psi + \frac{\partial U}{\partial \beta}\cos\phi\cos\psi + \frac{\partial U}{\partial \gamma}.0, \quad \ldots \quad (2\ 43)$$

$$\frac{\partial U}{\partial \psi} = -\frac{\partial U}{\partial \alpha}\cos\phi\sin\psi - \frac{\partial U}{\partial \beta}\sin\phi\sin\psi + \frac{\partial U}{\partial \gamma}\cos\psi, \quad (2\ 44)$$

and from Euler's formula

$$nU = \frac{\partial U}{\partial \alpha}\cos\phi\cos\psi + \frac{\partial U}{\partial \beta}\sin\phi\cos\psi + \frac{\partial U}{\partial \gamma}\sin\psi \quad \ldots \quad (2\ 45)$$

Solving equations (2 43), (2 44), (2 45) for $\frac{\partial U}{\partial \alpha}, \frac{\partial U}{\partial \beta}, \frac{\partial U}{\partial \gamma}$ we have,

$$\frac{\partial U}{\partial \alpha} = -\frac{\sin\phi}{\cos\psi}\frac{\partial U}{\partial \phi} - \cos\phi\sin\psi\frac{\partial U}{\partial \psi} + n\cos\phi\cos\psi\, U, \quad (2\ 51)$$

$$\frac{\partial U}{\partial \beta} = \frac{\cos\phi}{\cos\psi}\frac{\partial U}{\partial \phi} - \sin\phi\sin\psi\frac{\partial U}{\partial \psi} + n\sin\phi\cos\psi\, \mathbf{U}, \quad (2\ 52)$$

$$\frac{\partial U}{\partial \gamma} = 0\,\frac{\partial U}{\partial \phi} + \cos\psi\frac{\partial U}{\partial \psi} + n\sin\psi\, U. \quad (2\ 53)$$

Thus if $\frac{\partial}{\partial \alpha}$ operates on a homogeneous function of the *nth* degree in α, β, γ then we have the operational identity

$$\frac{\partial}{\partial \alpha} = -\frac{\sin\phi}{\cos\psi}\frac{\partial}{\partial \phi} - \cos\phi\sin\psi\frac{\partial}{\partial \psi} + n\cos\phi\cos\psi, \quad (2\ 54)$$

and similarly for $\frac{\partial}{\partial \beta}$ and $\frac{\partial}{\partial \gamma}$.

It is thus seen that the expression for $\frac{\partial}{\partial \alpha}$ in terms of $\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \psi}$ contains integer n. Therefore $\frac{\partial}{\partial \alpha}$ as operating on a homogeneous function of the *nth* degree in α, β, γ, when looked upon as built up of $\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial \psi}$, can be conveniently called $\frac{\partial}{\partial \alpha_{(n)}}$, and we can write

$$\frac{\partial}{\partial \alpha_{(n)}} = -\frac{\sin\phi}{\cos\psi}\frac{\partial}{\partial\phi} - \cos\phi\sin\psi\frac{\partial}{\partial\psi} + n\cos\phi\cos\psi, \qquad (2.55)$$

$$\frac{\partial}{\partial \beta_{(n)}} = \frac{\cos\phi}{\cos\psi}\frac{\partial}{\partial\phi} - \sin\phi\sin\psi\frac{\partial}{\partial\psi} + n\sin\phi\cos\psi, \quad \ldots \quad (2.56)$$

$$\frac{\partial}{\partial \gamma_{(n)}} = 0 + \cos\psi\frac{\partial}{\partial\psi} + n\sin\psi. \quad \ldots \quad (2.57)$$

3. From (2.55), (2.56), (2.57) it is readily seen that

$$\frac{\partial}{\partial \alpha_{(n)}} = \text{grad}_x + n\alpha, \quad \ldots \quad (2.61)$$

where $\text{grad}_x U$ represents the x-component of the vector grad U, and similarly

$$\frac{\partial}{\partial \beta_{(n)}} = \text{grad}_y + n\beta, \quad \ldots \quad (2.62)$$

$$\frac{\partial}{\partial \gamma_{(n)}} = \text{grad}_z + n\gamma. \quad \ldots \quad (2.63)$$

Remembering that H is of the first degree in α, β, γ

$$H_1 = \frac{\partial}{\partial \alpha_{(1)}} H = \text{grad}_x H + \alpha H, \quad \ldots \quad (2.64)$$

$$H_2 = \frac{\partial}{\partial \beta_{(1)}} H = \text{grad}_y H + \beta H, \quad \ldots \quad (2.65)$$

$$H_3 = \frac{\partial}{\partial \gamma_{(1)}} H = \text{grad}_z H + \gamma H. \quad \ldots \quad (2.66)$$

$$\therefore H_1^2 + H_2^2 + H_3^2 = \text{grad } H \cdot \text{grad } H + H^2$$

$$= \nabla H + H^2 \quad \ldots \quad (2.71)$$

Again

$$H_1\frac{\partial}{\partial \alpha_{(n)}} + H_2\frac{\partial}{\partial \beta_{(n)}} + H_3\frac{\partial}{\partial \gamma_{(n)}} = (\text{grad}_x + \alpha H)\left(\frac{\partial}{\partial \alpha_{(0)}} + n\alpha\right)$$

+ two similar terms

$$=\left(\operatorname{grad}_x \mathrm{H}\ \frac{\partial}{\partial \alpha_{(0)}}+n\alpha \operatorname{grad}_x \mathrm{H}+\mathrm{H}\alpha\frac{\partial}{\partial \alpha_{(0)}}+n\mathrm{H}\alpha^2\right)$$

$$+ \text{ two similar terms.}$$

Now by Euler's theorem

$$\alpha\frac{\partial}{\partial \alpha_{(0)}}+\beta\frac{\partial}{\partial \beta_{(0)}}+\gamma\frac{\partial}{\partial \gamma_{(0)}}=0,$$

while $$\alpha \operatorname{grad}_x \mathrm{H}+\beta \operatorname{grad}_y \mathrm{H}+\gamma \operatorname{grad}_z \mathrm{H}=0.$$

as grad H lies in the tangent plane to the sphere. Hence

$$\mathrm{H}_1\frac{\partial}{\partial \alpha_{(n)}}+\mathrm{H}_2\frac{\partial}{\partial \beta_{(n)}}+\mathrm{H}_3\frac{\partial}{\partial \gamma_{(n)}}=\operatorname{grad}\mathrm{H}.\operatorname{grad}+n^2\mathrm{H}. \quad \ldots \ (2.72)$$

4 All the integrations in this paper, unless otherwise stated, should be understood to be taken over the whole of the unit sphere. The double sign of integration will always for shortness be replaced by a single sign. We shall now prove an important lemma

Let U be a function of position on the unit sphere being homogeneous in α, β, γ and of degree n When U is regarded as a function of ψ and ϕ we can replace

$$\frac{\partial \mathrm{U}}{\partial \alpha} \text{ by } \frac{\partial}{\partial \alpha_{(n)}}\mathrm{U}.$$

Hence

$$\mathrm{I}=\int\frac{\partial \mathrm{U}}{\partial \alpha}d\omega$$

$$=\int\frac{\partial}{\partial \alpha_{(n)}}\mathrm{U}d\omega$$

$$=\int\!\!\left(-\frac{\sin\phi}{\cos\psi}\frac{\partial \mathrm{U}}{\partial \phi}-\cos\phi\sin\psi\frac{\partial \mathrm{U}}{\partial \psi}\right.$$

$$\left.+n\mathrm{U}\cos\phi\cos\psi\right)\cos\psi d\phi d\psi \text{ from } (2.55)$$

On integrating by parts, the first portion with respect to ϕ, the second portion with respect to ψ, and leaving the third portion

unchanged, and noting that since the integration is over the whole of the unit sphere, the partially integrated parts vanish, we have,

$$\begin{aligned} \mathrm{I} &= \int (\cos\phi + \cos\phi\cos 2\psi + n\cos\phi\cos^2\psi)\mathrm{U}\,d\phi d\psi \\ &= (n+2)\int \mathrm{U}\cos\phi\cos^2\psi\, d\phi d\psi \\ &= (n+2)\int \alpha\mathrm{U}\,d\omega, \quad \text{from } (2\ 11) \end{aligned}$$

We have thus shown that

$$\left.\begin{aligned} \int \frac{\partial \mathrm{U}}{\partial \alpha}\,d\omega &= (n+2)\int \alpha\mathrm{U}\,d\omega, \\ \int \frac{\partial \mathrm{U}}{\partial \beta}\,d\omega &= (n+2)\int \beta\mathrm{U}\,d\omega, \\ \int \frac{\partial \mathrm{U}}{\partial \gamma}\,d\omega &= (n+2)\int \gamma\mathrm{U}\,d\omega \end{aligned}\right\} \quad \ldots \quad (2.8)$$

If we replace U by VW, where V and W are homogeneous functions of α, β, γ of degrees m and n respectively we get the formulae in a slightly different form

$$\left.\begin{aligned} \int \mathrm{V}\frac{\partial \mathrm{W}}{\partial \alpha}\,d\omega &= -\int \mathrm{W}\frac{\partial \mathrm{V}}{\partial \alpha}\,d\omega + (m+n+2)\int \alpha\mathrm{VW}\,d\omega, \\ \int \mathrm{V}\frac{\partial \mathrm{W}}{\partial \beta}\,d\omega &= -\int \mathrm{W}\frac{\partial \mathrm{V}}{\partial \beta}\,d\omega + (m+n+2)\int \beta\mathrm{VW}\,d\omega, \\ \int \mathrm{V}\frac{\partial \mathrm{W}}{\partial \gamma}\,d\omega &= -\int \mathrm{W}\frac{\partial \mathrm{V}}{\partial \gamma}\,d\omega + (m+n+2)\int \gamma\mathrm{VW}\,d\omega, \end{aligned}\right\} \quad \ldots \quad (2.9)$$

The formulae (2.8) and (2.9) will be extremely useful to us and will be called the fundamental formulae for integration by parts.

III

The Gaussian curvature centroid

If to every point P of our closed convex surface Ω, we associate a density equal to the Gaussian curvature, then we can define the centroid of the surface so weighted, to be the *Gaussian curvature centroid of* Ω

Let x^0, y^0, z^0 be the rectangular co-ordinates of the Gaussian curvature centroid. Then

$$z^0 = \int \frac{1}{RR'} z\,dS \Big/ \int \frac{1}{RR'} dS, \tag{3.1}$$

where dS is the element of the surface, and the integration extends over the surface. Since

$$dS = RR'd\omega$$

this gives us

$$z^0 \int d\omega = \int z\,d\omega$$

$$= \int \frac{\partial \Pi}{\partial \gamma} d\omega \text{ from (2.21)}$$

In formula (2.8), putting $U = \Pi$ we have (since $n=1$)

$$\int \frac{\partial \Pi}{\partial \gamma} d\omega = 3\int \gamma \Pi d\omega.$$

Therefore
$$4\pi z^0 = 3\int \gamma \Pi d\omega \quad \ldots \quad (3.21)$$

Likewise
$$4\pi x^0 = 3\int \alpha \Pi d\omega, \quad \ldots \quad (3.22)$$

$$4\pi y^0 = 3\int \beta \Pi d\omega. \quad \ldots \quad (3.23)$$

Hence if $\mathbf{r}^0$ *denotes the vector from the origin to the Gaussian curvature centroid*

$$4\pi \mathbf{r}^0 = 3\int \Pi \mathbf{n}\,d\omega \quad \ldots \quad (3.3)$$

where $\mathbf{n}$ *is the unit vector normal to the unit sphere over which we are integrating.*

IV

The mean curvature centroid

If to every point of our closed convex surface Ω we associate a density equal to the mean curvature

$$\tfrac{1}{2}\left(\frac{1}{R}+\frac{1}{R'}\right)$$

then the centroid of the surface so weighted may be called the *mean curvature centroid* of Ω

Let x', y', z' be the rectangular co-ordinates of the mean curvature centroid. Then

$$z'=\int \tfrac{1}{2}\left(\frac{1}{R}+\frac{1}{R'}\right)z\,dS\bigg/\int \tfrac{1}{2}\left(\frac{1}{R}+\frac{1}{R'}\right)dS \qquad \dots \quad (4.11).$$

Therefore
$$M_0 z'=\int(R+R')z\,d\omega, \qquad \dots \quad (4.12)$$

where

$$M_0=\tfrac{1}{2}\int\left(\frac{1}{R}+\frac{1}{R'}\right)dS=\tfrac{1}{2}\int(R+R')d\omega \qquad \dots \quad (4.2)$$

Substituting from (2.21) and (2.31) in (4.12) we have

$$M_0 z'=\tfrac{1}{2}\int\frac{\partial H}{\partial\gamma}(H_{11}+H_{22}+H_{33})d\omega \qquad \dots \quad (4.3)$$

$$=\tfrac{1}{2}\int H_3\frac{\partial}{\partial\alpha}H_1 d\omega+\tfrac{1}{2}\int H_3\frac{\partial}{\partial\beta}H_2 d\omega$$

$$+\tfrac{1}{4}\int\frac{\partial}{\partial\gamma}(H_3^2)d\omega.$$

Applying the fundamental formula (2.9) to the three integrals in the above expression, and remembering that H_1, H_2, H_3 are of zero degrees in α, β, γ we have

$$M_0 z'=-\tfrac{1}{2}\int H_1H_{13}d\omega+\int\alpha H_1H_3 d\omega-\tfrac{1}{2}\int H_2H_{23}d\omega+\int\beta H_2H_3 d\omega$$

$$+ \tfrac{1}{2}\int \gamma H_3^2 d\omega$$

$$= - \tfrac{1}{2}\int \frac{\partial}{\partial \gamma} (H_1^2) d\omega + \int \alpha H_1 H_3 d\omega$$

$$- \tfrac{1}{2}\int \frac{\partial}{\partial \gamma} (H_2^2) d\omega + \int \beta H_2 H_3 d\omega$$

$$+ \tfrac{1}{2}\int \gamma H_3^2 d\omega$$

Again applying (2 8) we have

$$M_0 z' = - \tfrac{1}{2}\int \gamma H_1^2 d\omega + \int \alpha H_1 H_3 d\omega - \tfrac{1}{2}\int \gamma H_2^2 d\omega + \int \beta H_2 H_3 d\omega$$

$$+ \tfrac{1}{2}\int \gamma H_3^2 d\omega$$

$$= - \tfrac{1}{2}\int \gamma (H_1^2 + H_2^2 + H_3^2) d\omega + \int H_3 (\alpha H_1 + \beta H_2 + \gamma H_3) d\omega$$

or from (2 22)

$$M_0 z' = - \tfrac{1}{2}\int \gamma (H_1^2 + H_2^2 + H_3^2) d\omega + \int H H_3 d\omega$$

$$= - \tfrac{1}{2}\int \gamma (H_1^2 + H_2^2 + H_3^2) d\omega + \tfrac{1}{2}\int \frac{\partial}{\partial \gamma} (H^2) d\omega$$

$$= - \tfrac{1}{2}\int \gamma (H_1^2 + H_2^2 + H_3^2) d\omega + 2\int \gamma H^2 d\omega \quad \text{from (2.8)}.$$

Hence from (2.71)

$$2M_0 z' = \int (3H^2 - \nabla H)\gamma d\omega \qquad \dots \quad (4.41)$$

Likewise

$$2M_0 x' = \int (3H^2 - \nabla H)\alpha d\omega \qquad \dots \quad (4.42)$$

$$2M_0 y' = \int (3H^2 - \nabla H)\beta d\omega. \qquad \dots \quad (4.43)$$

If $\mathbf{r}'$ denotes the vector from the origin to the mean curvature centroid we can write

$$2M\mathbf{r}' = \int (3H^2 - \nabla H)\mathbf{n}\,d\omega, \qquad \ldots \quad (4.5)$$

where as before $\mathbf{n}$ is the unit vector normal to the unit sphere

V

The surface centroid.

The centroid of a closed convex surface Ω when a uniform density is supposed to be associated with every point of Ω, may be called the *surface centroid* of Ω.

Let x'', y'', z'' be the rectangular co-ordinates of the surface centroid. Then

$$z'' = \int z\,dS \Big/ \int dS \qquad \ldots \quad (5.1)$$

Therefore $\quad S_0 z'' = \int z R R' d\omega$, where S_0 is the surface area of Ω

$$= \int H_3 \frac{H_{22}H_{33} - H_{23}^2}{a^2}\,d\omega, \text{ from (2.21) and (2.35)}$$

$$= \tfrac{1}{2}\int \frac{1}{a^2} H_{33} \frac{\partial}{\partial \gamma}(H_3^2)\,d\omega$$

$$- \tfrac{1}{2}\int \frac{1}{a^2} H_{23} \frac{\partial}{\partial \beta}(H_3^2)\,d\omega$$

$$= -\tfrac{1}{2}\int \frac{\gamma}{a^2} H_3^2 H_{22}\,d\omega + \tfrac{1}{2}\int \frac{\beta}{a^2} H_{23} H_3^2\,d\omega \qquad \text{from (2.9)}$$

$$= \int \frac{\gamma}{a^2} H_2 H_3 H_{23}\,d\omega - \tfrac{1}{2}\int \frac{\beta\gamma}{a^2} H_2 H_3^2\,d\omega$$

$$- \tfrac{1}{6}\int H_3^3\,d\omega - \tfrac{1}{6}\int \frac{\gamma^2}{a^2} H_3^3\,d\omega \qquad \text{from (2.9) again.}$$

Substituting from (2 22) in the first of these integrals we have

$$S_0{}^{2''} = -\int \frac{H_2 H_3 H_{12}}{\alpha} d\omega - \int \frac{\beta}{\alpha^2} H_2 H_3 H_{22} d\omega$$

$$- \tfrac{1}{2}\int \frac{\beta\gamma}{\alpha^3} H_2 H_3^2 d\omega - \tfrac{1}{3}\int H_3^3 d\omega - \tfrac{1}{6}\int \frac{\gamma^2}{\alpha^2} H_3^3 d\omega \quad \ldots \quad (5.2)$$

Now the second of these integrals by two applications of the fundamental formulae, and the use of the relation (2 12), can be written as

$$\tfrac{1}{3}\int \frac{\beta\gamma}{\alpha^2} H_2^3 d\omega + \tfrac{1}{2}\int H_2^2 H_3 d\omega + \tfrac{1}{2}\int \frac{\gamma^2}{\alpha^2} H_2^2 H_3 d\omega$$

$$= \tfrac{1}{3}\int \frac{\beta\gamma}{\alpha} H_2^2 H_{12} d\omega - \tfrac{1}{3}\int \beta\gamma H_2^3 d\omega + \tfrac{1}{2}\int H_2^2 H_3 d\omega$$

$$+ \tfrac{1}{2}\int \frac{\gamma^2}{\alpha}(2H_2 H_3 H_{12} + H_2^2 H_{13}) d\omega - \tfrac{1}{2}\int \gamma^2 H_2^2 H_3 d\omega, \quad (5.31)$$

on integrating the first and third terms with regard to α

Likewise the third integral in (5.2), on integration with respect to α, gives

$$- \tfrac{1}{2}\int \frac{\beta\gamma}{\alpha}(H_3^2 H_{12} + 2H_2 H_3 H_{13}) d\omega + \tfrac{1}{2}\int \beta\gamma H_2 H_3^2 d\omega \quad (5.32)$$

and the fifth integral in (5.2) gives

$$- \tfrac{1}{2}\int \frac{\gamma^2}{\alpha} H_3^2 H_{13} d\omega + \tfrac{1}{6}\int \gamma^2 H_3^3 d\omega. \quad \ldots \quad (5.33)$$

Substituting in (5.2) from (5 31), (5 32) and (5 33) and collecting only the terms in $1/\alpha$ we have

$$\tfrac{1}{2}\int \Big\{ \frac{\beta\gamma}{\alpha} H_2^2 H_{12} + \frac{\gamma^2}{\alpha}(2H_2 H_3 H_{12} + H_2^2 H_{13})$$

$$- \frac{\gamma}{\alpha}(H_3^2 H_{12} + 2H_2 H_3 H_{13}) - \frac{\gamma^2}{\alpha} H_3^2 H_{13} - \frac{2}{\alpha} H_2 H_3 H_{12} \Big\} d\omega$$

$$= \tfrac{1}{2}\int(\gamma H_2^2 H_{11} + \gamma H_3^2 H_{11} - 2\alpha H_2 H_3 H_{12} + 2\beta H_2 H_3 H_{11})\,d\omega \quad \ldots \quad (5.41)$$

$$= I_1 \text{ (say)},$$

on suitably arranging and substituting from (2.22)

Collecting the other terms we have

$$-\tfrac{1}{3}\int H_3^3 d\omega + \tfrac{1}{2}\int \beta\gamma H_3^2 d\omega + \tfrac{1}{2}\int (1-3\gamma^2) H_2^2 H_3 d\omega$$

$$+\ \tfrac{3}{2}\int \beta\gamma H_2 H_3^2 d\omega + \tfrac{1}{2}\int \gamma^2 H_3^2 d\omega = I_2 \text{ (say)} \quad \ldots \quad (5.42)$$

Then $$S_0 z_2'' = I_1 + I_2 \quad \ldots \quad (5.5)$$

where it should be noticed that I_2 is free from terms containing second differential co-efficients, and I_1 contains only such terms

The third term in I_1 is after substitution from (2 22) and integration with the help of the fundamental formula

$$-\tfrac{1}{2}\int \beta\gamma H_3^2 d\omega - \tfrac{1}{2}\int (1-3\beta^2) H_2^2 H_3 d\omega + \int \gamma H_2 H_3 H_{22} d\omega \ \ldots \ (5.61)$$

Substituting for βH_2 from (2 22) in the fourth term of I_1 and by repeated applications of the fundamental integration formulae we get,

$$-\tfrac{1}{2}\int H_1^2 H_3 d\omega - \tfrac{3}{2}\int \gamma H H_1^2 d\omega + 3\int \alpha H H_3 H_1 d\omega$$

$$+\ \tfrac{1}{2}\int \alpha\gamma H_1^2 d\omega + \tfrac{1}{2}\int (1-3\alpha^2) H_1^2 H_3 d\omega - \int \gamma H_3^2 H_{11} d\omega \ \ldots \ (5.62)$$

Noting that the last term in (5.62) taken together with the first two terms of I_1, gives on integration

$$\int \gamma(H_1 H_2 H_{12} + H_3 H_1 H_{13})\,d\omega - \tfrac{3}{2}\int \alpha\gamma H_1 (H_2^2 + H_3^2)\,d\omega \ \ldots \ (5.63)$$

we have on collecting all the terms

$$S_0\epsilon'' = \int\gamma(H_2H_3H_{23}+H_1H_2H_{12}+H_3H_1H_{13})d\omega$$

$$-\ \tfrac{1}{6}\int(1-3\gamma^2)H_1^2d\omega + \tfrac{1}{2}\int\alpha\gamma H_1^2 d\omega + \tfrac{3}{2}\int(\beta^2-\gamma^2)H_2^2H_3d\omega$$

$$+\ \tfrac{3}{2}\int\beta\gamma H_2H_1^2d\omega - \tfrac{3}{2}\int\alpha^2H_1^2H_3d\omega - \tfrac{3}{2}\int\alpha\gamma H_3^2H_1d\omega$$

$$-\ \tfrac{3}{2}\int\alpha\gamma H_1H_2^2d\omega - \tfrac{3}{2}\int\gamma HH_1^2d\omega + 3\int\alpha HH_1H_3d\omega \qquad (5\cdot71)$$

Interchanging α and β in (5·71) we get another similar formula for $S_0\epsilon''$. On adding up these two and halving we have

$$S_0\epsilon'' = \int\gamma(H_2H_3H_{23}+H_1H_2H_{12}+H_3H_1H_{31})d\omega$$

$$-\ \tfrac{1}{6}\int(1-3\gamma^2)H_3^2d\omega + \tfrac{1}{4}\int\gamma(\alpha H_1^2+\beta H_2^2)d\omega$$

$$-\ \tfrac{3}{4}\int\gamma^2H_3(H_2^2+H_1^2)d\omega - \tfrac{3}{4}\int\gamma(\alpha H_1H_3^2+\beta H_3^2H_2)d\omega$$

$$-\ \tfrac{3}{4}\int\gamma H(H_1^2+H_2^2)d\omega + \tfrac{3}{2}\int HH_3(\alpha H_1+\beta H_2)d\omega \quad \dots \quad (5\cdot72)$$

$$= \int\gamma(H_2H_3H_{23}+H_1H_2H_{12}+H_3H_1H_{31})d\omega$$

$$-\ \tfrac{1}{6}\int(1-3\gamma^2)H_3^2d\omega + \int\gamma(\alpha H_1^2+\beta H_2^2)d\omega$$

$$-\ \tfrac{3}{4}\int\gamma H(H_1^2+H_2^2+H_3^2)d\omega + \tfrac{3}{2}\int\gamma H^3d\omega \qquad (5\cdot73)$$

on substitution from (2.22), simplification and integration of one term.

The first three integrals in (5 73) can be easily shown to reduce to

$$\tfrac{1}{4}\int \gamma\left(H_1 \frac{\partial}{\partial \alpha} + H_2 \frac{\partial}{\partial \beta} + H_3 \frac{\partial}{\partial \gamma} \right)(H_1^2+H_2^2+H_3^2)d\omega \quad \ldots \quad (5.74)$$

Therefore we have finally

$$S_0 z'' = \tfrac{1}{4}\int\gamma \left(H_1 \frac{\partial}{\partial \alpha} + H_2 \frac{\partial}{\partial \beta} + H_3 \frac{\partial}{\partial \gamma} \right)(H_1^2+H_2^2+H_3^2)d\omega$$

$$- \tfrac{3}{2}\int\gamma H(H_1^2+H_2^2+H_3^2)d\omega + \tfrac{3}{2}\int\gamma H^3 d\omega$$

$$= \tfrac{1}{4}\int\gamma(\text{grad } H \ \text{grad})(\nabla H+H^2)d\omega$$

$$- \tfrac{3}{2}\int\gamma H(\nabla H+H^2)d\omega + \tfrac{3}{2}\int\gamma H^3 d\omega$$

from (2 71) and (2.72), remembering that $H_1^2+H_2^2+H_3^2$ is of zero degree in α, β, γ

$$\text{Therefore } S_0 z'' = \int\{H^3 - H\nabla H + \tfrac{1}{4}(\text{grad } H \ \text{grad})\nabla H\}\gamma d\omega \qquad (5\ 8)$$

with similar formulae for $S_0 x''$, $S_0 y''$.

If r'' be the vector from the origin to the surface centroid, and n the unit vector normal to the unit sphere, we have

$$S_0 r'' = \int\{H^3 - H\nabla H + \tfrac{1}{4}(\text{grad } H.\ \text{grad})\nabla H\}n d\omega \qquad (5.9)$$

VI.

The volume centroid.

The centroid of the volume, enclosed by the closed convex surface Ω, a uniform density being supposed to be associated with each point, may be called the *volume centroid* of Ω.

Let x''', y''', z''' be the rectangular co-ordinates of the volume centroid Then

$$z''' = \tfrac{3}{4}\cdot\tfrac{1}{3}\int \mathrm{H}z\,d\mathrm{S} \Big/ \tfrac{1}{3}\int \mathrm{H}\,d\mathrm{S} \qquad \ldots \quad (6.1)$$

where the integrations are taken over the whole surface

Therefore $\quad 2\mathrm{V}_0 z''' = \tfrac{1}{2}\int \mathrm{H}\mathrm{H}_3 \mathrm{R}\mathrm{R}'\,d\omega$

where V_0 is the volume enclosed by Ω

Therefore $\quad 2\mathrm{V}_0 z''' = \tfrac{1}{2}\int \mathrm{H}\mathrm{H}_3 \dfrac{\mathrm{H}_{22}\mathrm{H}_{33} - \mathrm{H}_{23}^2}{a^2}\,d\omega \qquad$ from (2.35)

$$= -\tfrac{1}{6}\int \frac{1}{a^2}\,\mathrm{H}_3^3 \mathrm{H}_{22}\,d\omega + \tfrac{1}{3}\int \frac{1}{a^2}\,\mathrm{H}_3^2 \mathrm{H}_2 \mathrm{H}_{23}\,d\omega$$

$$= \int \frac{1}{a^2}\mathrm{H}_2 \mathrm{H}_3^2 \mathrm{H}_{23}\,d\omega$$

$$= \int \left(1 + \frac{\beta^2}{a^2} + \frac{\gamma^2}{a^2}\right) \mathrm{H}_2 \mathrm{H}_3^2 \mathrm{H}_{23}\,d\omega. \qquad \ldots \quad (6.2)$$

This integral can be evaluated by exactly the same method as we used in the previous section, consisting in the repeated application of the fundamental integration formulae (2.8), (2.9) together with a judicious application of the identities (2.22). We get in this way

$$2\mathrm{V}_0 z''' = \tfrac{1}{4}\int (\mathrm{H}_3 + \gamma\mathrm{H})\left(\mathrm{H}_1 \frac{\partial}{\partial a} + \mathrm{H}_2 \frac{\partial}{\partial \beta} + \mathrm{H}_3 \frac{\partial}{\partial \gamma}\right)(\mathrm{H}_1^2 + \mathrm{H}_2^2 + \mathrm{H}_3^2)\,d\omega$$

$$+ \tfrac{1}{4}\int \gamma(\mathrm{H}_1^2 + \mathrm{H}_2^2 + \mathrm{H}_3^2)^2\,d\omega$$

$$- \int (2\gamma\mathrm{H}^2 + \tfrac{1}{2}\,\mathrm{H}\mathrm{H}_3)(\mathrm{H}_1^2 + \mathrm{H}_2^2 + \mathrm{H}_3^2)\,d\omega + 3\int \gamma\mathrm{H}^4\,d\omega$$

$$= \tfrac{1}{4}\int (\mathrm{H}_3 + \gamma\mathrm{H})(\mathrm{grad}\ \mathrm{H}.\ \mathrm{grad})(\nabla\mathrm{H} + \mathrm{H}^2)\,d\omega$$

$$+\ \tfrac{1}{2}\int\gamma(\nabla H+H^2)^2 d\omega\ -\int(2\gamma H^2+\tfrac{1}{2}HH_3)(\nabla H+H^2)d\omega\ +\ 3\int\gamma H^4 d\omega$$

from (2 71) and (2.72)

$$=\ \tfrac{1}{2}\int H_3\ (\text{grad } H.\ \text{grad})\nabla H d\omega$$

$$+\ \tfrac{1}{4}\int\gamma\{(\nabla H)^2+H^4+2H^2\nabla H\}d\omega$$

$$+\ \tfrac{1}{2}\int\gamma\{H\ (\text{grad } H.\ \text{grad})\ \Delta H\}d\omega$$

$$-\ \int\gamma H^2\nabla H d\omega\ +\ \tfrac{1}{2}\int\gamma H_4 d\omega$$

Using now (2 66) we have

$$4V_0 z'''=\ \tfrac{1}{2}\int(\text{grad}, H)(\text{grad } H\ \text{grad})(\nabla H)d\omega\ +\ \tfrac{1}{2}\int\gamma\{H^4-2H^2\nabla H$$

$$+\ H\ (\text{grad } H\ \text{grad})(\nabla H)+\tfrac{1}{2}(\nabla H)^2\}d\omega \qquad \ldots \quad (6\ 3)$$

with similar formulae for $4V_0 x'''$ and $4V_0 y'''$.

If r''' be the vector from the origin to the volume centroid, and n the unit vector normal to the unit sphere, we have

$$4V_0 r'''=\ \int\{H^4-2H^2\nabla H+H\ (\text{grad } H\ \text{grad})(\nabla H)+\tfrac{1}{2}(\nabla H)^2\}n d\omega$$

$$+\ \tfrac{1}{2}\int\{(\text{grad } H\ \text{grad})(\nabla H)\}\ \text{grad } H d\omega \qquad \ldots \quad (6.4)$$

VII.

The three invariants of the surface

If M_0 denotes the integral of the mean curvature, taken over the whole convex surface, S_0 the surface area, and V_0 the volume enclosed, then M_0, S_0 and V_0 may be called the three invariants of the surface. Then

$$M_0=\int\frac{1}{2}\left(\frac{1}{R}+\frac{1}{R'}\right)dS$$

where the integration extends over the whole surface

Therefore
$$M_0 = \tfrac{1}{2}\int(R+R')d\omega$$

$$= \tfrac{1}{2}\int(\Pi_{11}+\Pi_{22}+\Pi_{33})d\omega \qquad \text{from (2.31)}$$

$$= \int(\alpha\Pi_1+\beta\Pi_2+\gamma\Pi_3)d\omega \qquad \text{from (2.8)}$$

$$= \int \Pi d\omega \qquad \text{from (2.22)}$$

In the same way, we can after some calculation prove the known formula ... (7.1)

$$S_0=\int (\Pi^2-\tfrac{1}{2}\nabla\Pi)d\omega \qquad \text{... (7.2)}$$

We propose to give in slightly greater detail, the derivation of a similar formula for V_0, which is believed to be new,

$$V_0 = \tfrac{1}{3}\int \Pi dS,$$

where the integration is over the whole surface

Therefore
$$V_0 = \tfrac{1}{3}\int \Pi RR'd\omega$$

$$= \tfrac{1}{3}\int \frac{\Pi(\Pi_{11}\Pi_{22}-\Pi_{12}^2)}{\gamma^2}d\omega \qquad \text{from (2.37)}$$

$$= \tfrac{1}{3}\int \frac{\Pi_1^2\Pi_{22}-\Pi_1\Pi_2\Pi_{12}}{\gamma^2}d\omega$$

$$= \int \frac{\Pi_1\Pi_2\Pi_{12}}{\gamma^2}d\omega$$

$$= \int\left(1+\frac{\beta^2}{\gamma^2}+\frac{\alpha^2}{\gamma^2}\right)\Pi_1\Pi_2\Pi_{12}d\omega.$$

By the use of previous methods this finally leads to

$$V_0 = \tfrac{1}{4}\int\left(H_1\frac{\partial}{\partial\alpha}+H_2\frac{\partial}{\partial\beta}+H_3\frac{\partial}{\partial\gamma}\right)(H_1^2+H_2^2+H_3^2)d\omega$$

$$-\int H(H_1^2+H_2^2+H^2)d\omega+\tfrac{4}{3}\int H^3 d\omega$$

$$=\tfrac{1}{4}\int(\text{grad } H \ \text{grad})(\nabla H+H^2)d\omega$$

$$-\int H(\nabla H+H^2)d\omega+\tfrac{4}{3}\int H^3 d\omega \qquad \text{from (2.71) and (2.72)}$$

Hence finally

$$V_0=\int\{\tfrac{1}{3}H^3-\tfrac{1}{2}H\nabla H+\tfrac{1}{4}(\text{grad } H, \text{grad})(\nabla H)\}d\omega. \quad \dots \quad (7.3)$$

Substituting from (7.1), (7.2) and (7.3) in (4.5), (5.9) and (6.4) we get the vectors r', r'', r''' purely in terms of the tangential function (Stutzfunktion) H. Formula (2.3) already expresses r^0 in terms of H.

VIII

The two curvature centroids of a surface of constant breadth.

A convex surface is called a *surface of constant breadth* when the distance between parallel tangent planes is constant For such a surface

$$H(\alpha, \beta, \gamma)+H(-\alpha, -\beta, -\gamma)=D=\text{a const.} \quad . \quad (8.1)$$

Our integrations have so far been performed over the whole uni sphere. We shall now denote by

$$\int_{\Omega} d\omega \qquad \dots \quad (8.2)$$

integration over a hemisphere of the unit sphere, which lies to th positive side of the xy plane. If as before (x^0, y^0, z^0), $(x', y', z'$

denote the co-ordinates of Gaussian and the mean curvature centroids then

$$z^0 = \frac{3}{4\pi}\int \gamma \mathrm{H} d\omega$$

$$= \frac{3}{4\pi}\int_{\Omega}\{\gamma\mathrm{H}-\gamma(\mathrm{D}-\mathrm{H})\}d\omega$$

$$= \frac{3}{2\pi}\int_{\Omega}\gamma\mathrm{H}d\omega - \tfrac{3}{4}\mathrm{D} \qquad \ldots \quad (8{,}3)$$

$$2\mathrm{M}_0 z' = \int(3\mathrm{H}^2 - \nabla\mathrm{H})\gamma d\omega$$

$$= \int_{\Omega}\gamma\{3\mathrm{H}^2-\nabla\mathrm{H}-3(\mathrm{D}-\mathrm{H})^2+\nabla\mathrm{H}\}d\omega$$

$$= \int_{\Omega}\gamma(6\mathrm{DH}-3\mathrm{D}^2)d\omega. \qquad \ldots \quad (8{\cdot}4)$$

Again from (7.1)

$$\mathrm{M}_0 = \int \mathrm{H} d\omega$$

$$= \int_{\Omega}\{\mathrm{H}+(\mathrm{D}-\mathrm{H})\}d\omega$$

$$= 2\pi\mathrm{D}. \qquad \ldots \quad (8{\cdot}5)$$

Hence from (8·4) and (8,5) we have

$$z' = \frac{3}{2\pi}\int_{\Omega}\gamma\mathrm{H}d\omega - \tfrac{3}{4}\mathrm{D} \qquad \ldots \quad (8{,}6)$$

Therefore $\qquad z^0 = z'.$

By integrating over suitably chosen hemispheres we can similarly show that

$$x^0 = x' \text{ and } y^0 = y'.$$

Thus *for a surface of constant breadth, the Gaussian and the mean curvature centroids coincide.* This corresponds to Meissner's theorem, that *the curvature centroid of an oval of constant breadth coincides with it's perimeter centroid.*

IX

The loci of the four centroids for a system of parallel convex surfaces.

Let Ω_t denote the convex surface parallel to the given convex surface Ω, at a distance t from it, t being measured positively along the outward normal. If ρ denotes the lower bound of the minimum radii of curvatures of Ω, then if t ranges from $-\rho$ to ∞, Ω_t will still be a convex surface. Let $\boldsymbol{r}^0(t)$, $\boldsymbol{r}'(t)$ $\boldsymbol{r}''(t)$, $\boldsymbol{r}'''(t)$ denote the vectors from the origin to the Gaussian curvature centroid, the mean curvature centroid, the surface centroid, and the volume centroid of Ω_t. Let $V=V(t)$, $S=S(t)$ and $M=M(t)$, denote the volume, the surface, and the integral of mean curvature for Ω_t. Then clearly

$$V(0)=V_0,\ S(0)=S_0,\ M(0)=M_0. \qquad \dots \quad (9.11)$$

From (3.3)

$$\boldsymbol{r}^0(t)=\frac{3}{4\pi}\int(H+t)\boldsymbol{n}\,d\omega$$

$$=\frac{3}{4\pi}\int H\boldsymbol{n}\,d\omega$$

$$=\boldsymbol{r}^0.$$

Again from (4.5) we have

$$M\boldsymbol{r}'(t)=\tfrac{1}{2}\int\{3(H+t)^2-\nabla(H+t)\}\boldsymbol{n}\,d\omega$$

$$=\tfrac{1}{2}\int(3H^2-\nabla H)\boldsymbol{n}\,d\omega+3t\int H\boldsymbol{n}\,d\omega$$

$$=M_0\boldsymbol{r}'+4\pi t\boldsymbol{r}^0. \qquad \dots \quad (9.13)$$

In the same way from (5.9) we get

$$S\boldsymbol{r}''(t)=S_0\boldsymbol{r}''+2M_0t\boldsymbol{r}'+4\pi t^2\boldsymbol{r}^0 \qquad \dots \quad (9.14)$$

and from (6.4) we get

$$V\boldsymbol{r}'''(t)=V_0\boldsymbol{r}'''+S_0t\boldsymbol{r}''+M_0t^2\boldsymbol{r}'+\frac{4\pi}{3}t^3\boldsymbol{r}^0 \qquad \dots \quad (9.15)$$

Again from (7.1), (7.2), (7.3) we easily have

$$V=V_0+S_0t+M_0t^2+\frac{4\pi}{3}t^3 \qquad \dots \quad (9.21)$$

$$S=S_0+2M_0t+4\pi t^2 \qquad \dots \quad (9.22)$$

$$M=M_0+4\pi t. \qquad \dots \quad (9.23)$$

The results (9.21) to (9.23) are originally due to Steiner.*

Hence finally we have

$$r^0(t) = r^0 \qquad \dots \quad (9.31)$$

$$r'(t) = \frac{M_0 r' + 4\pi t r^0}{M_0 + 4\pi t} \qquad \dots \quad (9.32)$$

$$r''(t) = \frac{S_0 r'' + 2M_0 t r' + 4\pi t^2 r^0}{S_0 + 2M_0 t + 4\pi t^2} \qquad \dots \quad (9.33)$$

$$r'''(t) = \frac{V_0 r''' + S_0 t r'' + M_0 t^2 r' + \frac{4\pi}{3} t^3 r^0}{V_0 + S_0 t + M_0 t^2 + \frac{4\pi}{3} t^3}. \qquad \dots \quad (9.34)$$

Equations (9.31) to (9.34) are the vector equations of the loci of the four centroids, for the system of parallel surfaces. We thus see:

The Gaussian curvature centroid of a system of parallel convex surfaces is a fixed point G^0, the locus of G', the mean curvature centroid is a straight line Σ', the locus of G'', the surface centroid is a conic section Σ'', while the locus of G''', the volume centroid is a rational space cubic Σ'''.

That the locus of G'' is a conic has been otherwise proved by Kubota.†

When $t = \infty$,

$$r'(t) = r''(t) = r'''(t) = r^0 \qquad \dots \quad (9.4)$$

This shows that $\Sigma', \Sigma'', \Sigma'''$ all pass through G^0.

The equations (9.21) to (9.23) readily show that

$$S = \frac{dV}{dt} \qquad \dots \quad (9.51)$$

$$2M = \frac{d^2V}{dt^2} \qquad \dots \quad (9.52)$$

$$8\pi = \frac{d^3V}{dt^3} \qquad \dots \quad (9.53)$$

* Steiner: Über parallele Flächen, Gesammelte Werke II pp. 173-176.

† Kubota: *Loc. cit.*

Again if we take a vector function $\xi = \xi(t)$ given by

$$\xi = V_0 r''' + S_0 t r'' + M_0 t^2 r' + \frac{4\pi}{3} t^3 r^0 \qquad \ldots \quad (9.54)$$

then

$$\frac{d\xi}{dt} = S_0 r'' + 2M_0 t r' + 4\pi t^2 r^0 \qquad \ldots \quad (9.55)$$

$$\frac{d^2\xi}{dt^2} = 2(M_0 r' + 4\pi t r^0) \qquad \ldots \quad (9.56)$$

$$\frac{d^3\xi}{dt^3} = 8\pi \qquad \ldots \quad (9.57)$$

Substituting from (9.51) to (9.57) in (9.12), (9.13), (9.14), (9.15), we have

$$r^0 \frac{d^3V}{dt^3} = \frac{d^3\xi}{dt^3} \qquad \ldots \quad (9.61)$$

$$r'(t) \frac{d^2V}{dt^2} = \frac{d^2\xi}{dt^2} \qquad \ldots \quad (9.62)$$

$$r''(t) \frac{dV}{dt} = \frac{d\xi}{dt} \qquad \ldots \quad (9.63)$$

$$r'''(t) V = \xi \qquad \ldots \quad (9.64)$$

From (9.64), we have

$$V \frac{d}{dt} r'''(t) + \frac{dV}{dt} r'''(t) = \frac{d\xi}{dt} = \frac{dV}{dt} r''(t) \quad \text{from (9.63)}$$

Hence
$$V \frac{d}{dt} r'''(t) = \frac{dV}{dt} \{r''(t) - r'''(t)\}, \qquad \ldots \quad (9.65)$$

Similarly
$$\frac{dV}{dt} \frac{d}{dt} r''(t) = \frac{d^2V}{dt^2} \{r'(t) - r''(t)\} \qquad \ldots \quad (9.66)$$

and
$$\frac{d^2V}{dt^2} \frac{d}{dt} r'(t) = \frac{d^3V}{dt^3} \{r'(t) - r^0(t)\} \qquad \ldots \quad (9.67)$$

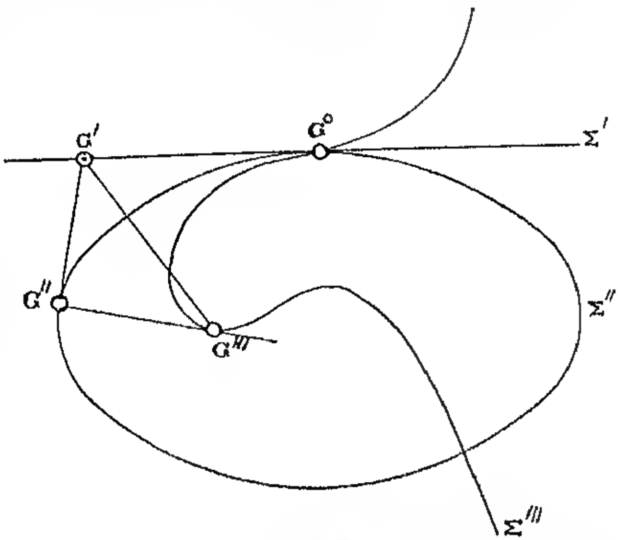
Again from (9.65)

$$V \frac{d^2}{dt^2} r'''(t) = \frac{dV}{dt} \left\{ \frac{d}{dt} r''(t) - 2\frac{d}{dt} r'''(t) \right\} + \frac{d^2V}{dt^2} \{r''(t) - r'''(t)\}$$

$$= \frac{d^2V}{dt^2} \{r'(t) - r'''(t)\} - \frac{2}{V} \left(\frac{dV}{dt} \right)^2 \{r''(t) - r'''(t)\} \qquad \ldots \quad (9.68)$$

Let now G_0, G', G'', G''' denote the four centroids of the parallel convex surface Ω_t, at a distance t from Ω. Then

$$\frac{d}{dt}\, r''(t)$$



is a vector parallel to the tangent at G'' to the conic Σ'', which is the locus of G''. Also $r'(t)-r''(t)$ is the vector $G''G'$. The relation (9.66) thus shows that the tangent to Σ'' at G'', passes through G'. In the same way the relation (9.65), shows that the tangent at G''' to the space cubic Σ''', which is the locus of G''', must pass through G''. Again the relation (9.68) shows, that the vectors

$$\frac{d^2}{dt^2}\, r'''(t),\; r'(t)-r''(t) \text{ and } r''(t)-r'''(t)$$

are coplanar. But the first vector is parallel to the principal normal at G''' to Σ''', while the other two vectors are $G''G'$ and $G'''G''$. Thus the osculating plane at G''' to Σ''' passes through G'. We may also state the same thing by saying that $G'\,G''\,G'''$ is the osculating plane to Σ''' at G'''.

Again consider the three vectors $\lambda(t)$, $\mu(t)$ and $\nu(t)$ defined by

$$\lambda(t) \equiv (M_0+4\pi t)\{r'(t)-r''(t)\}$$

$$= \frac{M_0S_0(r'-r'')+4\pi S_0(r^0-r'')t+4\pi M_0(r^0-r')t^2}{S_0+2M_0t+4\pi t^2}, \quad \ldots \quad (9.71)$$

from (9.32) and (9.33),

$$\mu(t) \equiv t\{r''(t)-r'''(t)\}$$

$$= \frac{A}{B} \qquad \ldots \quad (9.72)$$

where $\quad A \equiv V_0S_0(r''-r''')t+2V_0M_0(r'-r''')t^2$
$$+4\pi V_0(r^0-r''')t^3+M_0S_0(r'-r'')t^3$$

$$+\frac{8\pi}{3}S_0(r^0-r'')t^4+\frac{4\pi}{3}M_0(r^0-r')t^5$$

$$B \equiv (S_0+2M_0t+4\pi t^2)(V_0+S_0t+M_0t^2+\frac{4\pi}{3}t^3)$$

$$\nu(t) \equiv 3\left\{\frac{V_0+S_0t+M_0t^2+(4\pi/3)t^3}{M_0+4\pi t}\right\}\{r''(t)-(r'''(t)\}$$
$$-t^2\{r'(t)-r''(t)\}$$

$$=\frac{C}{D} \qquad \text{.. (9 73)}$$

where $C \equiv 3V_0S_0(r''-r''')+6V_0M_0(r'-r''')t$
$+12\,\pi V_0(r^0-r''')t^2+2M_0S_0(r'-r'')t^2$
$+4\pi S_0(r^0-r'')t^3$

and $D \equiv (M_0+4\pi t)(S_0+2M_0t+4\pi t^2)$

Now the vector $\lambda(t)$ is a scalar multiple of the vector $G''G'''$, and hence (from what has been already proved) is parallel to the tangent at G'' to Σ'' But for $t\longrightarrow\infty$ the point $G''\longrightarrow G^0$, while $\lambda(t)\longrightarrow r^0-r'$, which is the vector $G'G^0$. This shows that the conic Σ'' touches the line Σ' at G^0. The relation (9 72) similarly shows that the line Σ' is a tangent to the space cubic Σ''' at G^0. Finally $\nu(t)$ is coplanar with the vectors $G'''G''$ and $G''G'$, and is therefore parallel to the osculating plane at G''' to Σ'''. But for $t\longrightarrow\infty$, the point $G'''\longrightarrow G^0$, while $\nu(t)\longrightarrow r^0-r''$. Hence the vector $G''G^0$ is parallel to the osculating plane of Σ''' at G^0. As we have already shown $G'G^0$ is a tangent to Σ''' at G^0, it follows that the osculating plane of Σ''' at G^0 coincides with the plane $G^0G'G''$, which is the plane of the conic Σ''.

Summing up our results we can say :—

If Ω is any regular analytic closed convex surface, and if we construct a series of convex surfaces parallel to Ω, then the Gaussian curvature centroid G^0 remains fixed, the locus of the mean curvature centroid is a line Σ' passing through G^0, the locus of the surface centroid is a conic Σ'' touching the line Σ' at G^0, and the locus of the volume centroid is a rational space cubic Σ''' touching Σ' at G^0, and having the plane of the conic Σ'' as its osculating plane at G^0 If G^0, G', G'', G''' be the four centroids for any of these surfaces, then the tangent at G'' to Σ'' passes through G', the tangent at G''' to Σ''' passes through G'', while the osculating plane at G''' to Σ''', coincides with the plane $G'G''G'''$.

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A Reduction-formula for the Functions of the second kind connected with the Polynomials of Applied Mathematics

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(*Communicated by the Editorial Secretary*)

1. *Introductory.*

In course of an investigation regarding rotating fluids, I have had occasion to make considerable use of Legendre's function of the second kind, which, as is well known, may be expressed in the form

$$Q_n(z) = \tfrac{1}{2}P_n(z)\log\frac{z+1}{z-1} - f_{n-1}(z),$$

where P_n is Legendre's polynomial, and f_{n-1} a polynomial of degree $n-1$. Various expressions of f_{n-1} have been given, particularly in terms of Legendre's polynomials of lower degrees; but I found that none was satisfactory in regard to the practical computations which I had to perform. I tried, therefore, to obtain a suitable form; and, introducing a polynomial $B_n(z)$, of degree $n-1$, such that

$$A_n(z)P_n(z) + B_n(z)P_n'(z) \equiv 1,$$

where A_n is a polynomial of degree $n-2$, I found the following remarkable expression:—

$$f_{n-1}(z) = \frac{zP_n(z)-B_n(z)}{z^2-1}.$$

The same process allowed me in a similar way to obtain a simple reduction-formula for Lamé's function of the second kind. I then

formed the idea of extending this method of the treatment of other functions, and the result of this investigation is the theme of the present paper.

2. *The General Method*

Let us consider a polynomial $f_n(z)$, of degree n, satisfying the differential equation of the second order

$$a(z)\frac{d^2f_n(z)}{dz^2} + b(z)\frac{df_n(z)}{dz} + c(z)f_n(z) = 0 \quad .. \quad \text{(E)}$$

where a, b, c are polynomials.

We shall assume that the two following conditions are satisfied .

1. All the roots of $f_n(z)$ are simple
2. None of them is a root of $a(z)$

We shall remark that these conditions are satisfied for nearly all the polynomials of Mathematical Physics

We shall write

$$\frac{b(z)}{a(z)} = -\frac{g'(z)}{g(z)}$$

A second solution of equation (E), known as the *function of the second kind* connected with f_n, can be immediately written under Euler's form ·

$$q_n(z) = f_n(z)\int^z \frac{g(t)}{[f_n(t)]^2}\,dt, \quad \dots \quad (1)$$

the lower limit of the integral being conveniently chosen

We shall now introduce two polynomials $A_n(z)$ and $B_n(z)$, of degrees respectively lower than those of f_n' and f_n, and such that

$$A_n(z)f_n(z) + B_n(z)f_n'(z) \equiv 1, \quad \dots \quad (2)$$

a property sometimes described as *Bezout's Formula*.

We can then write,

$$\frac{q_n(z)}{f_n(z)} = \int\left(\frac{A_n}{f_n} + \frac{B_nf_n'}{f_n^2}\right)g\,dt$$

Integrating by parts, and supposing that $\frac{B_n g}{f_n}$ vanishes at the lower limit of the integral, we obtain

$$\frac{q_n(z)}{f_n(z)} = -\frac{B_n(z)g(z)}{f_n(z)} + \int\left(\frac{A_n B_n'}{f_n} + \frac{B_n g'}{f_n g}\right)g\,dt$$

We shall write the quantity between brackets

$$\frac{A_n + B_n'}{f_n} - \frac{B_n b}{f_n a},$$

and try to find a new expression for it.

From identity (2) we have

$$A_n f_n' + A_n' f_n + B_n' f_n' + B_n f_n'' = 0, \qquad \ldots \quad (3)$$

whence $$\frac{A_n + B_n'}{f_n} = -\frac{A_n'}{f_n'} - \frac{B_n f_n''}{f_n f_n'}$$

Denoting now by $\mu_1, \mu_2, \ldots\ldots \mu_n$ the roots of f_n and by $\nu_1, \nu_2, \ldots\ldots \nu_{n-1}$ the roots of f_n' we have

$$\frac{A_n'}{f_n'} = \sum_{i=1}^{n-1} \frac{A_n'(\nu_i)}{f_n''(\nu_i)(z-\nu_i)}$$

$$\frac{B_n f_n''}{f_n f_n'} = \sum_{i=1}^{n-1} \frac{B_n(\mu_i) f_n''(\mu_i)}{f_n'^2(\mu_i)(z-\mu_i)} + \sum_{i=1}^{n-1} \frac{B_n(\nu_i)}{f_n(\nu_i)(z-\nu_i)}.$$

But from (3)

$$\frac{A_n'(\nu_i)}{f_n''(\nu_i)} + \frac{B_n(\nu_i)}{f_n(\nu_i)} = 0,$$

and so

$$\frac{A_n + B_n'}{f_n} = -\sum_{i=1}^{n} \frac{B_n(\mu_i) f_n''(\mu_i)}{f_n'^2(\mu_i)(z-\mu_i)}.$$

Now from (E) we have

$$\frac{f_n''(\mu_i)}{f_n'(\mu_i)} = -\frac{b(\mu_i)}{a(\mu_i)}$$

whence $$\frac{A_n+B_n'}{f_n} = \sum_{i=1}^{n} \frac{B_n(\mu_i)b(\mu_i)}{a(\mu_i)f_n'(\mu_i)} \cdot \frac{1}{z-\mu_i}.$$

We can also write

$$\frac{B_n b}{af_n} = \sum_{i=1}^{n} \frac{B_n(\mu_i)b(\mu_i)}{a(\mu_i)f_n'(\mu_i)(z-\mu_i)} + \sum_{i=1}^{m} \frac{B_n(s_i)b(s_i)}{a'(s_i)f_n(s_i)(z-s_i)} \quad (4)$$

where $s_1, s_2, .. s_m$ are the roots of $a(\cdot)$

[We suppose here that the degree of $b(z)$ is inferior or equal to the degree of $a(\cdot)$, which is generally the case in the equations of Applied Mathematics, we shall, however, see afterwards an example of the contrary, introducing into this equality a constant term]

We thus obtain

$$\frac{A_n+B_n'}{f_n} - \frac{B_n b}{af_n} = -\sum_{i=1}^{m} \frac{B_n(s_i)b(s_i)}{a'(s_i)f_n(s_i)(z-s_i)},$$

and, using the formula, we obtain for q_n the required expression —

$$q_n(z) = -B_n(z)g(z) - f_n(z)\sum_{i=1}^{m} \frac{B_n(s_i)b(s_i)}{a'(s_i)f_n(s_i)} \int^{2} \frac{g(t)dt}{t-s_i}. \quad ... \quad (5)$$

We shall now discuss this result.—

(a) Comparing this value of q_n with the expression given by (1), we see that it contains m integrals instead of one, but, firstly, these m integrals are similar in form, and secondly, not containing f_n, they are far more simple than the integral in (1)

(b) The polynomials of mathematical physics are generally solutions of differential equations where $a(z)$ and $b(z)$ are independent of the degree n, which appears only in $c(z)$ The m integrals in (5) are therefore independent of n and appear in the expression of the function q of any order So, if $m<n$, we can reduce q_n to the m functions $q_1, q_2, . q_m$, and our formula (5) will be a general reduction-formula for q_n We must note that the function q_0 cannot be introduced there, for, f_0 being a constant, the polynomial B_0 is nugatory. This reduction-formula seems to be specially suitable for the numerical computations which often occur in harmonic analysis and allied questions.

(c) We shall presently establish a general formula for the B_n's which will be of great help in simplifying our expression (5)

Let us suppose that *the degree of* $b(z) <$ *the degree of* $a(z)$, and consider again the equality (4) Owing to the fact that

$$B_n(\mu_i) f_n'(\mu_i) = 1,$$

it can be written

$$\frac{B_n b}{a f_n} = \sum_{i=1}^{m} \frac{B_n(s_i) b(s_i)}{a'(s_i) f_n(s_i)(z-s_i)} - \sum_{i=1}^{n} \frac{f_n''(\mu_i)}{f_n'^2(\mu_i)(z-\mu_i)}.$$

Let us multiply the two members by z and let z become infinite, observing now that

$$\Sigma \frac{f_n''(\mu_i)}{f_n'^2(\mu_i)} = 0,$$

as being the sum of the residues for the rational fraction $\dfrac{1}{f_n^2(z)}$, we obtain the required formula

$$\sum_{i=1}^{m} \frac{B_n(s_i) b(s_i)}{a'(s_i) f_n(s_i)} = 0$$

(d) Suppose now that we have to deal with polynomials which are solutions of a differential equation of the hypergeometric type: this is a very important case We are led to two integrals; but using the preceding formula, we shall reduce them to one integral only, independent of n and so the function q_n will be reduced to q_1.

(e) It is not necessary to remark that the polynomial $B_n(z)$ is rather easy to form; it can be obtained from $f_n(z)$ by rational operations only, the knowledge of the roots of $f_n(z)$ is not required In many cases, as we shall see hereafter, it is possible to form a recurrence formula between the B's and f's, which will be of great help in obtaining readily the values of $B_n(s_i)$

We shall illustrate this general theory by various examples, chosen as follows;

1 $a(z)$ is of degree 3. Lamé's equation,

2 $a(z)$ is of degree 2; and $b(z)$ of degree 1; Legendre's equation and its extension.

3 $a(z)$ and $b(z)$ are each of degree 1; Laguerre's equation.

4. $a(z)$ is of degree 0, $b(z)$ of degree 1. Hermite's equation.

3. *Lamé's Functions.*

We shall not give here the details of the treatment of Lamé's functions of the second kind by the above method, as this development is to appear elsewhere, together with its application to the theory of Poincaré's figures of equilibrium for a rotating mass of fluid. We shall only state the following results.

When Lamé's function of the first kind is a polynomial, *i.e.*, when n is even, there is no difficulty at all, and the function of the second kind S_n can readily be reduced to functions S of the first and the second order.

When n is odd, Lamé's function being no longer a polynomial, but the product of a polynomial T_n by an irrational factor, the method fails. However, it can be applied with certain modifications, taking for A_n and B_n the polynomials connected with T_n by Bezout's formula, and introducing in the first course of the work, double roots for the decomposition of rational fractions, a formula analogous to the preceding though a little more complicated, can be obtained, which allows the reduction of S_n to the two functions S of order 1 and 2

4. *The Extended Legendre's Polynomials.*

These functions studied by Gegenbauer, furnish us with a good example of our method. Let us first briefly recall some of their properties *

The polynomial $C_n^{\nu}(z)$ is defined by the expansion

$$(1-2az+a^2)^{-\nu} = \sum_{n=1}^{\infty} a^n C_n^{\nu}(z).$$

It satisfies the differential equation

$$(1-z^2)y'' - (2\nu+1)zy' + n(2\nu+n)y = 0.$$

It can be expressed in terms of Gauss's hypergeometric function by the formula:

$$C_n^{\nu}(z) = \frac{\Gamma(n+2\nu)}{\Gamma(n+1)\Gamma(2\nu)} F\left(-n,\ n+2\nu,\ \nu+\tfrac{1}{2}\ ;\ \frac{1-z}{2}\right)$$

* For the properties, expansions, recurrence formulae, etc., of the function the reader is referred to Whittaker and Watson, Modern Analysis, 323 See also Appell and Lambert, Généralisation des fonctions sphériques (Édition Française de Encyclopédie, ii, 5), 237

Amongst the recurrence formulae of this function we shall use the two following

$$nC_n^\nu = (n+2\nu-1)zC_{n-1}^\nu+(z^2-1)C_{n-1}^{\nu'} \quad \dots \quad \text{(I)}$$

$$nzC_n^\nu = (n+2\nu-1)C_{n-1}^\nu+(z^2-1)C_n^{\nu'}. \quad \dots \quad \text{(II)}$$

With C_n^ν is associated a function of the second kind, which we will write as

$$H_n^\nu = C_n^\nu(z)\int_\infty^z \frac{dt}{(1-t^2)^{\nu+\frac{1}{2}}[C_n^\nu(t)]^2} \,.$$

The general method may be applied without any difficulty; we shall introduce the polynomials $B_n^\nu(z)$, and in this particular case formula (6) will be written as

$$\frac{B_n^\nu(1)}{C_n^\nu(1)} + \frac{B_n^\nu(-1)}{C_n^\nu(+1)} = 0$$

In order to evaluate $B_n^\nu(1)$, we shall establish the following recurrence formula:

$$(\mu+2\nu-1)C_{n-1}^{\nu'}(z)B_n^\nu(z) - nC_n^\nu(z)B_{n-1}^\nu(z) + z^2-1 = 0.$$

The demonstration is easy. we have a polynomial of degree $2n-2$; and we shall prove that it has $2n-1$ roots, *viz.*, the $n-1$ roots $\alpha_1, \alpha_2 \dots\dots \alpha_{n-1}$ of $C_{n-1}^\nu(z)$ and the n roots of $\beta_1 \dots\dots \beta_n$ of $C_n^\nu(z)$. For a root α_i, it becomes

$$-nC_n^\nu(\alpha_i) + (\alpha_i^2-1)\, C_{n-1}^{\nu'}(\alpha_i),$$

since

$$B_{n-1}^\nu(\alpha_i)C_{n-1}^{\nu'}(\alpha_i) = 1.$$

It is therefore zero by recurrence formula I. For a root β_i it becomes

$$(n+2\nu-1)C_{n-1}^\nu(\beta_i) + (\beta_i^2-1)C_n^{\nu'}(\beta_i)$$

which is zero by II. Then, since

$$C_n^\nu(1) = \frac{\Gamma(n+2\nu)}{\Gamma(n+1)\Gamma(2\nu)},$$

our formula gives us

$$B_n^\nu(1) = B_{n-1}^\nu(1).$$

But $C_1^\nu(z)$ being $2\nu z$, $B_1^\nu(z)$ is $\frac{1}{2\nu}$, so

$$B_n^\nu(1) = \frac{1}{2\nu}$$

and

$$\frac{B_n^\nu(1)}{C_n^\nu(1)} = \frac{\Gamma(n+1)\Gamma(2\nu)}{2\nu\Gamma(n+2\nu)}.$$

The general method gives us the following result

$$H_n^\nu(z) = -\frac{B_n^\nu(z)}{(1-z^2)^{\nu+\frac{1}{2}}} + \frac{(2\nu+1)\Gamma(n+1)\Gamma(2\nu)}{2\nu\Gamma(n+2\nu)} C_n^\nu(z) \int \frac{dt}{(1-t^2)^{\nu+\frac{3}{2}}}.$$

Two forms can then be proposed: we can, as we have said, express the above integral in terms of $H_0^\nu(z)$, and write

$$H_n^\nu(z) = -\frac{B_n^\nu(z)}{(1-z^2)^{\nu+\frac{1}{2}}} + \frac{(2\nu+1)\Gamma(n+1)\Gamma(2\nu)}{2\nu\Gamma(n+2\nu)} C_n^\nu(z) H_0^{\nu+1}(z).$$

These are two reduction formulae for H_n^ν.

For $\nu=\frac{1}{2}$, we have the ordinary Legendre's functions; the first formula becomes

$$Q_n(z) = -\frac{B_n(z)}{1-z^2} + \frac{P_n(z)}{z(1-z^2)} + \frac{P_n(z)}{z} Q_1(z).$$

As $Q_1(z)$ is $\frac{1}{2}z \log \frac{z-1}{z-1} - 1$,

this can be written

$$Q_n(z) = \frac{zP_n(z)-B_n(z)}{1-z^2} + \frac{1}{2}P_n(z) \log \frac{z+1}{z-1},$$

which is the formula we spoke of at the begining of the paper.

Professor Whittaker having suggested that the function B_n might be put in the form of a determinant, I have found for B_n^ν the following expression as a determinant with $(n-1)$ rows:

$$B_n^\nu = \frac{1}{2\nu(2\nu+1)\dots(2\nu+n-1)} \begin{vmatrix} (2\nu+1)z & +(2\nu+1) & 0 & 0 & 0 \\ 2 & (2\nu+1)z & 2\nu+2 & 0 & 0 \\ 0 & 3 & (2\nu+6)z & 2\nu+3 & 0 \\ 0 & 0 & 4 & (2\nu+8)z & 2\nu+4 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

This expression includes as a particular case, the determinantal form for the function B_n connected with Legendre's polynomials.

5 *Laguerre's Polynomials*

The polynomial L_n of Laguerre * is defined by

$$L_n(z) = e^{-z} \frac{d^n}{dz^n}(z^n e^z)$$

$$zL_n''(z) + (z+1)L_n'(z) - nL_n(z) = 0$$

The function of the second kind connected with this polynomial is

$$M_n(z) = L_n(z)\int_{\infty}^{z} \frac{e^{-t}\,dt}{t[L_n(t)]^2}.$$

We do not develop this case, which is extremely simple. A recurrence formula can readily be obtained between the B's and L's, it may be proved that $L_n(z)=n!$, and $B_n(0)=\frac{1}{n!}$; and we obtain

$$[n!]^2 M_n(z) = e^{-z}\pi_{n-1}(z) + L_n(z)M_0(z)$$

where $\pi_{n-1}(\cdot)$ is the polynomial

$$\tfrac{1}{2}[L_n(z)-(n!)^2 B_n(z)],$$

and it may be noted that $M_0(z)$ is nothing else than the well-known logarithmic integral function, $li(z)$.

6. *Hermite's Polynomial* † *or the Parabolic Cylinder function when n is an integer.*

The Parabolic Cylinder † function when n is an integer, degenerates into the product of an exponential and a polynomial;

$$D_n(z) = e^{-\frac{z^2}{4}} U_n(z).$$

U_n is Hermite's polynomial, which satisfies the equation

$$U_n'' - zU_n' + nU_n = 0$$

* Laguerre, Oeuvres, I, 428. Appell and Lambert, *op. cit.*, 280. These polynomials can be expressed in terms of the confluent hypergeometric function

† Whittaker, Proc. Lond. Math. Soc., **XXXV**, Arch. Milne, *op. cit.*, xxxii,

A second solution is given by

$$V_n(z) = U_n(z)\int_0^z \frac{e^{\frac{t^2}{2}}dt}{[U_n(t)]^2}.$$

We shall consider only the case when n is an even number. Then $U_n(0)=1$, $B_n(0)=0$, and it may be proved that

$$n(n-1)U_{n-2}(z)B_n(z) - U_n(z)B_{n-2}(z) - z = 0.$$

Now if we proceed with the general method, we shall have to consider the decomposition of the rational factor $\frac{zB_n(z)}{U_n(z)}$; and a constant term must be introduced. In that case, we are led to write

$$\frac{A_n+B_n'}{U_n} + \frac{zB_n}{U_n} = K_n,$$

$$K_n = \lim_{z=\infty} \frac{zB_n(z)}{U_n(z)}.$$

The recurrence formula between B and U may be used to prove that

$$K_n = \frac{1}{n!},$$

and finally we write

$$V_n(z) = -e^{\frac{z^2}{2}}B_n(z) + \frac{1}{n!}U_n(z)V_0(z).$$

A second solution of the differential equation of the parabolic cylinder when n is an even number can then be written

$$\Delta_n(z) = -B_n(z)e^{\frac{z^2}{4}} + \frac{1}{n!}D_n(z)\Delta_0(z)$$

The functions V_0 or Δ_0 may be reduced, by a slight change of variable to the error-function $\mathrm{Erf}(z)$.

These examples will suffice to show how this general method must be used for the reduction of functions of the second kind.

Other functions to which a similar treatment may be applied are the Jacobean polynomials $F_n(z)$, which can be written $F(-n, \alpha+n; \nu, z)$ and confluent hypergeometric function $W_{k,m}(z)$ when $m-k$ is of the form $n+\frac{1}{2}$, being an integer.

Bull, Cal. Math. Soc., Vol. XXVII, Nos. 3 & 4 (1935).

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NOTE ON THE STABILITY OF A THIN PLATE UNDER EDGE THRUST, BUCKLING BEING RESISTED BY A SMALL FORCE VARYING AS THE DISPLACEMENT

BY

B. SEN

1 *Introduction and the statement of problem*

The problem of stability of a thin plate under uniform edge thrust was first solved by G. H. Bryan.* Since then buckling under different conditions has been considered by several investigators.† On account of the physical importance of the problem, it is thought desirable to consider the stability of a thin plate under uniform edge thrust when buckling is resisted by a small force proportional to the displacement.

We assume

the thickness of the plate $=2h$,

the edge thrust per unit of length $=2hP$,

the small deflection of the plate $=w$,

the flexural rigidity of the plate $=D$ $=\dfrac{2Eh^3}{3(1-\sigma^2)}$,

and the resistance per unit of area $=cw$. ... (1.1)

Then the equation of equilibrium in the slightly deflected position

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$$D\nabla_1^4 w+2hP\nabla_1^2 w=-cw,$$

or

$$\nabla_1^4 w+k^2\nabla_1^2 w+\lambda^2 w=0 \qquad \text{... (1.2)}$$

* Proceedings of the London Mathematical Society (Ser. 1), Vol. 22 (1891), p. 54.

† 'The Mathematical Theory of Elasticity,' by A. E. H. Love, 4th edition, p. 538.

where $$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

$$k^2 = \frac{2hP}{D},$$

and $$\lambda^2 = \frac{c}{D} \text{ (a small quantity)} \qquad (1.3)$$

2 *A circular plate with clamped edge*

It is difficult to obtain directly the general solution of the equation

$$\nabla_1^4 w + k^2 \nabla_1^2 w + \lambda^2 w = 0, \qquad \dots \quad (2.1)$$

in polar co-ordinates However, particular values of w satisfying the equation can be obtained in an indirect manner We assume as the solution of (2.1)

$$w = AJ_0(\mu r), \qquad \dots \quad (2.2)$$

where J_0 is the Bessel's function of the first kind and zero order and A is a constant

Since

$$\nabla_1^2 w = -\mu^2 w,$$

and

$$\nabla_1^4 w = \mu^4 w$$

we obtain from (2.1)

$$(\mu^4 - k^2\mu^2 + \lambda^2) w = 0$$

or

$$(\mu^4 - k^2\mu^2 + \lambda^2) = 0 \qquad \dots \quad (2.3)$$

By hypothesis, λ^2 is very small and hence we can assume

$$k^4 > 4\lambda^2 \qquad \dots \quad (2.4)$$

Then two real values μ_1 and μ_2 of μ can be taken as

$$\mu_1 = \left[\frac{k^2 + \sqrt{k^4 - \lambda^2}}{2}\right]^{\frac{1}{2}},$$

and

$$\mu_2 = \left[\frac{k^2 - \sqrt{k^4 - \lambda^2}}{2}\right]^{\frac{1}{2}} \qquad \dots \quad (2.5)$$

the positive sign only being taken before the brackets as the values of Bessel's functions are the same whether the argument is positive or negative

Therefore we find that a solution of (2.1) can be put as

$$w = AJ_0(\mu_1 r) + BJ_0(\mu_2 r), \qquad \dots \quad (2.6)$$

A and B being constants

The boundary conditions for a clamped edge are

$$w=0 \text{ and } \frac{\partial w}{\partial r}=0 \quad \text{when } r=a. \qquad \dots \quad (2.7)$$

These conditions give

$$AJ_0(\mu_1 a)+BJ_0(\mu_2 a)=0,$$

and

$$A\mu_1 J'_0(\mu_1 a)+B\mu_2 J'_0(\mu_2 a)=0$$

Since

$$J'_0(x)=-J_1(x),$$

we have on eliminating A and B from the above

$$\mu_1 J_1(\mu_1 a)J_0(\mu_2 a)-\mu_2 J_1(\mu_2 a)J_0(\mu_1 a)=0 \qquad \dots \quad (2.8)$$

Let $\mu_1 a=\beta$ and $\frac{\mu_2}{\mu_1}=q$ (<1).

then

$$q=\frac{\mu_1\mu_2}{\mu_1^2}=\frac{\lambda}{\mu_1^2}=\frac{\lambda a^2}{\beta^2}. \qquad \dots \quad (2.9)$$

The equation (2.8) now becomes

$$J_1(\beta)J_0(q\beta)-qJ_1(q\beta)J_0(\beta)=0. \qquad \dots \quad (2.10)$$

Substituting the value of q obtained in (2.9) in the above equation, we get an equation in β, the roots of which give the possible values of k^2 and hence of P producing deflection. If λ be very small, we have

$$q=\frac{\lambda}{\mu_1^2}, \text{ also a small quantity.}$$

We can then assume for moderate values of β, $q\beta$ to be so small that the terms containing higher powers of $q\beta$ than the second may be neglected

Writing down the approximate values of $J_0(q\beta)$ and $J_1(q\beta)$, we get the equation (2.10) in the form

$$f(\beta)=J_1(\beta)-\frac{q^2\beta}{2^2}\left[\beta J_1(\beta)+2J_0(\beta)\right]=0 \qquad \dots \quad (2.11)$$

We find from the table that when $\beta=3\cdot 8317\ldots$

$$J_1(\beta)=0 \text{ and } J_0(\beta), \text{ a negative quantity.}$$

Hence for this value of β, $f(\beta)$ is positive.

Again when $\beta=5\cdot 5200\ldots$, we have

$$J_0(\beta)=0 \text{ and } J_1(\beta), \text{ a negative quantity.}$$

Since $\frac{q^2\beta^2}{2^2}$ is, by hypothesis, a small quantity less than unity, $f(\beta)$ is now negative

This shows that the deflection is possible for some value of

$$\beta \;\; (\text{that is, } \mu_1 a) \; > \; 3\cdot 8317\ldots$$

We have from (2 5) $\quad k^2>\mu_1^2$

whence we derive that the deflection is produced only when

$$\frac{2hP}{D}a^2 \; > \; (3\cdot 8317)^2\ldots$$

$$\text{i.e., when } 2hP > (3\cdot 8317\quad)^2\,\frac{D}{a^2}. \qquad \ldots \quad (2\cdot 12)$$

3. *A rectangular plate with supported edges*

The boundary conditions for a rectangular plate with supported edges can be written as

$$w=0, \text{ along the edges,}$$

$$\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}=0, \text{ along the sides } x=0, \text{ and } x=a,$$

$$\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}=0 \text{ along the sides } y=0, \text{ and } y=b \qquad (3.1)$$

All these conditions are satisfied if we take

$$w=A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \ldots \quad (3.2)$$

provided m and n are integers.

This expression for w will also satisfy the differential equation (1.2) if

$$\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right]^2-k^2\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right]+\lambda^2=0,$$

or if

$$k^2=\left[\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}}-\frac{\lambda}{\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}}}\right]^2+2\lambda \qquad (3.3)$$

Hence in order that the deflection may be possible, k^2 must be equal to or greater than 2λ.

That is,

$$h\Gamma = \text{ or } > \sqrt{c}\mathrm{D}. \qquad \dots \quad (3.3)$$

4. *A rectangular plate used as a strut*

In this case, we take the origin at the middle point of one of the sides and suppose that this edge and that parallel to it are supported while the other two edges are free. The length of each of the former pair is taken b and that of each of the other pair a. We further assume that there are no thrusts on the sides which are free.

The equation (1.2) in the present case reduces to

$$\dots \qquad \nabla_1^4 w+k^2\frac{\partial^2 w}{\partial x^2}+\lambda^2 w=0 \qquad \dots \quad (4.1)$$

The boundary conditions are

$$\left.\begin{aligned} &w=0,\\ &\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}=0 \end{aligned}\right\}, \text{ when } x=0 \text{ and } x=a, \qquad \dots \quad (4.2)$$

$$\left.\begin{aligned} &\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}=0\\ &\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0 \end{aligned}\right\}, \text{ when } y=\pm\frac{b}{2}, \qquad \dots \quad (4.3)$$

The conditions (4.2) are satisfied if we take

$$w = V \sin \frac{m\pi x}{a}, \qquad \ldots \quad (4.4)$$

V being a function of y and m an integer.

Substituting this value of w in (4.1), we get

$$\frac{\partial^4 V}{\partial y^4} - \frac{2\pi^2 m^2}{a^2} \frac{\partial^2 V}{\partial y^2} + \frac{\pi^4 m^4}{a^4} V = \left[\frac{k^2\pi^2 m^2}{a^2} \right] V = \alpha_V^4 \text{ (say)} . \qquad (4.5)$$

where $$\alpha^4 = \frac{k^2\pi^2 m^2}{a^2} - \lambda^4. \qquad \ldots \quad (4.6)$$

To solve the equation (4.5), we put

$$V = A \cosh qy \qquad \ldots \quad (4.7)$$

Then from (4.5), we have

$$\left(q^2 - \frac{\pi^2 m^2}{a^2} \right)^2 = \alpha^4,$$

from which we obtain four roots of the form $\pm q_1$, $\pm q_2$ satisfying the equations

$$q_1^2 - \frac{\pi^2 m^2}{a^2} = \alpha^2, \text{ and}$$

$$q_2^2 - \frac{\pi^2 m^2}{a^2} = -\alpha^2. \qquad \ldots \quad (4.8)$$

For satisfying the conditions (4.3), we take the symmetrical solution

$$V = A \cosh q_1 y + B \cosh q_2 y. \qquad \ldots \quad (4.9)$$

Then the boundary conditions give

$$A \cosh \frac{q_1 b}{2} \left[q_1^2 - \frac{\sigma\pi^2 m^2}{a^2} \right] + B \cosh \frac{q_2 b}{2} \left[q_2^2 - \frac{\sigma\pi^2 m^2}{a^2} \right] = 0 \qquad (4.10)$$

$$A \sinh \frac{q_1 b}{2} \left[q_1^2 - \frac{\pi^2 m^2}{a^2} \right] q_1 + B \sinh \frac{q_2 b}{2} \left[q_2^2 - \frac{\pi^2 m^2}{a^2} \right] q_2 = 0 \quad (4.11)$$

Eliminating A and B, we obtain with the help of the relation (4.8)

$$q_2\left[(2-\sigma)\frac{\pi^2m^2}{a^2}-q_2^2\right]\tanh\tfrac{1}{2}q_2b$$

$$=q_1\left[\frac{\sigma\pi^2m^2}{a^2}-q_2^2\right]\tanh\tfrac{1}{2}q_1b. \qquad \ldots \quad (4.12)$$

This equation, together with the relation

$$q_1^2=\frac{2\pi^2m^2}{a^2}-q_2^2,$$

will give the values of a and k satisfying the boundary conditions and the required differential equations.

Let us write

$$f(q_2)=q_2\left[(2-\sigma)\frac{\pi^2m^2}{a^2}-q_2^2\right]\tanh\tfrac{1}{2}q_2b$$

$$-q_1\left[\sigma\frac{\pi^2m^2}{a^2}-q_2^2\right]\tanh\tfrac{1}{2}q_1b. \qquad \ldots \quad (4.13)$$

When $q_2=0$, we find that $f(q_2)$ is negative, and when $q_2^2=\frac{\sigma\pi^2m^2}{a^2}$, $f(q_2)$ is positive.

Hence the equation (4.13) is satisfied for some value of q_2^2 lying between 0 and $\frac{\sigma\pi^2m^2}{a^2}$. For deflection to be possible for this value

$$q_2^2<\frac{\sigma\pi^2m^2}{a^2},$$

i.e., $$\frac{\pi^2m^2}{a^2}-\alpha^2<\frac{\sigma\pi^2m^2}{a^2},$$

i.e., $$\alpha^2>(1-\sigma)\frac{\pi^2m^2}{a^2},$$

i.e.,
$$\frac{k^2\pi^2m^2}{a^2}-\lambda^2 < (1-\sigma)^2\frac{\pi^4m^4}{a^4},$$

i.e.,
$$k^2 > (1-\sigma)^2\frac{\pi^2m^2}{a^2}+\frac{\lambda^2a^2}{\pi^2m^2},$$

or
$$> \left[(1-\sigma)\frac{\pi m}{a}-\frac{\lambda a}{\pi m}\right]^2+2\lambda(1-\sigma). \quad \ldots \quad (4\cdot14)$$

Hence unless k^2 be greater than $2\lambda(1-\sigma)$, there cannot be any deflection, for the value of q_2^2 lying between the abovementioned limits

Since $q_2^2 > 0$, we have

$$\frac{\pi^4m^4}{a^4} > a^4$$

i.e.,
$$> \frac{k^2\pi^2m^2}{a^2}-\lambda^2.$$

Hence
$$k^2 < \frac{\pi^2m^2}{a^2}+\frac{\lambda^2a^2}{\pi^2m^2}. \quad \ldots \quad (4\cdot15)$$

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On Infinite Integrals of Bessel Functions

N. G. Shabde

(*Nagpur*).

Introduction.—Some infinite integrals involving a product of Bessel functions in the integrand have been recently evaluated by Watson,* Bailey,† Rice‡ and others.§ The object of this note is to obtain certain more general integrals of the same type. We also note a particular case of a Meijer's result|| giving an integral expression in terms of Bessel functions for a product of two k_n functions.

§1.

We take the following expansion given by Bailey¶ :—

$$(1.1)\quad \begin{cases} \left(\frac{1}{2} z\right)^{k-\mu-\nu} J_\mu(az) J_\nu(bz) \\ = \dfrac{a^\mu b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \dfrac{(k+2n)\Gamma(k+n)}{n!} J_{k+2n}(z) \end{cases}$$

$$F_4[-n, k+n, \mu+1, \nu+1;\ a^2, b^2].$$

* *Journal London Math. Society*, **9**, Part 1, *10*, "An infinite integral involving Bessel functions."

† (i) *Proc. London Math. Soc.*, (Series 2), **40**, *37*; "Some infinite integrals involving Bessel functions." This paper will be referred to as B 1.

(ii) *Journal London Math. Soc.*, **11**, Part 1, *16*; "Some infinite integrals involving Bessel functions (II)." This will be referred to as B 2.

‡ *Quarterly Journal of Math.* (Oxford series), **6**, *52*, "On contour integrals for the product of two Bessel functions."

§ (i) *Journal of the Indian Math. Soc.*, (New series), **1**, *110*; "On some integrals involving Bessel functions," by M. Ziaud Din and N. G. Shabde.

(ii) *Proc. London Math. Soc.*, (Series 2), **40**, *1*; "Integraldarstellungen aus der Theorie der Besselschen Funktionen," by C. S. Meijer.

(iii) *Quarterly Journal of Math.* (Oxford series), **6**, *241*; "Einige Integraldarstellungen für Produkte von Whittakerschen Funktionen," by C. S. Meijer. This will be referred to as M.

|| See M, *241*.

¶ *Quarterly Journal of Math.* (Oxford Series), **6**, *233*; "Some expansions of Bessel functions involving Appell's function F_4." The formula quoted is (3.1).

To verify that this expansion is uniformly convergent in z we obtain a known integral by term-wise integration of (1.1)

Multiply (1.1) by $\dfrac{J_k(cz)}{z}$ and integrate both the sides from 0 to ∞ with respect to z. We get

$$(1.2)\qquad \int_0^\infty z^{\kappa-\mu-\nu-1} J_\mu(az)J_\nu(bz)J_\kappa(cz)dz$$

$$= \frac{a^\mu . b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^\infty \frac{(\kappa-2n)\Gamma(\kappa+n)}{n!} F_4[-n,\kappa+n, \mu+1,\nu+1, a^2,b^2]$$

$$\times \int_0^\infty \frac{J_k(cz)J_{k+2n}(z)}{z} dz.$$

The left-hand side of (1.2) becomes by B 1, formula (8 2), equal to

$$(1.3)\qquad \frac{2^{\kappa-\mu-\nu-1} a^\mu\, b^\nu \Gamma(\kappa)}{c^\kappa \Gamma(\mu+1)\Gamma(\nu+1)}.$$

The right hand side

$$= \sum_{n=0}^\infty \frac{a^\mu \cdot b^\nu . 2^{\kappa-\mu-\nu-1} \{\Gamma(\kappa-n)\}^2 2F_1\{n+1, n ; \kappa+2n+1, \frac{1}{c^2}\}.(\kappa+2n)}{c^{\kappa+2n}\Gamma(\kappa+2n+1)\Gamma(1-n)\quad n!}$$

by means of a formula of Sonine and Schafheitlin* is equal to (1.3) as all terms of the series except the first vanish because of the presence of $\Gamma(1-n)$ in the denominator.

The expansion in (1.1) being, thus, uniformly convergent, the term wise-integration in the following articles is justified.

* See Watson's Theory of Bessel functions, 1922, *401*, formula 2

§2.

To evaluate

$$\int_0^\infty e^{-ct} t^{\kappa+\lambda-\mu-\nu-1} J_\mu(at) J_\nu(bt) dt;$$

$$|c| > 1 \quad \textit{and} \quad R(\lambda+\kappa) > 0$$

From (1.1) we have

(2.1) $$\int_0^\infty e^{-ct}\, t^{\kappa+\lambda-\mu-\nu-1} J_\mu(at) J_\nu(bt) dt$$

$$= \frac{2^{\kappa-\mu-\nu} \cdot a^\mu\, b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\kappa+2n)\Gamma(\kappa+n)}{n!} F_4[-n, \kappa+n, \mu+1, \nu+1, a^2, b^2]$$

$$\times \int_0^\infty e^{-ct} J_{k+2n}(t)\, t^{\lambda-1} dt\ *$$

$$= \frac{2^{\kappa-\mu-\nu}\, a^\mu b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\kappa+2n)\Gamma(\kappa+n)}{n!} F_4[-n, \kappa+n;\mu+1, \nu+1, a^2, b^2]$$

$$\times \frac{\Gamma(\lambda+\kappa+2n).\ 2F_1\left[\frac{\kappa+\lambda+2n}{2}, \frac{1-\lambda+\kappa+2n}{2}, \kappa+2n+1, \frac{1}{1+c^2}\right]}{(c^2+1)^{\frac{1}{2}(\lambda+\kappa+2n)}\ 2^{\kappa+2n}\Gamma(\kappa+2n+1)}.$$

(2.2) $$\int_0^\infty J_\mu(at) J_\nu(bt) J_\rho(ct) J_\lambda(gt) t^{\kappa+\rho-\mu-\nu-1}\, dt$$

$$= \frac{a^\mu . b^\nu . 2^{\rho-\mu-\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \left[\int_0^\infty J_{\rho+2n}(t) J_\rho(ct) J_\lambda(gt)\, t^{k-1} dt\right]$$

$$\times \frac{(\rho+2n)\Gamma(\rho+n)}{n!} F_4[-n, \rho+n, \mu+1, \nu+1;\ a^2, b^2]$$

$$= \frac{2^{\kappa+\rho-\mu-\nu-1}\, a^\mu\, b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\rho+2n)\Gamma(\rho+n)}{n!}$$

$$\times \frac{F_4[-n, \rho+n;\ \mu+1, \nu+1, a^2, b^2]}{g^{\kappa+2\rho+2n}\ \Gamma(\rho+1)\Gamma(\rho+2n+1)} \times \frac{c^\rho\, \Gamma\{\frac{1}{2}(\kappa+2\rho+2n+\lambda)\}}{\Gamma\{1-\frac{1}{2}(\kappa+2\rho+2n-\lambda)\}}$$

$$\times F_4\left[\tfrac{1}{2}(\kappa+2\rho+2n-\lambda), \tfrac{1}{2}(\kappa+2\rho+2n+\lambda);\ \rho+2n+1;\ \rho+1, \frac{1}{g^2}, \frac{c^2}{g^2}\right]$$

by means of B1, p. 45, formula (7.1).

In (2.2), $g > c+1$, $R(\kappa) < \frac{3}{2}$ and $R\,(2\rho+\kappa+\lambda) > 0$.

* This integral can be evaluated by means of (3), 385 of Watson's Theory of Bessel functions

§3

The following integrals can also be similarly evaluated. We only write the final value in each case.

$$(3.1)\qquad \int_0^\infty J_\mu(at)J_\nu(bt)K_\rho(ct)J_\tau(gt)t^{\lambda+\kappa-\mu-\nu-1}\,dt$$

$$= \frac{2^{\kappa+\lambda-\mu-\nu-2}}{\Gamma(\mu+1)\Gamma(\nu+1)}\sum_{n=0}^\infty \frac{(k+2n)\Gamma(k+n)}{n!}\,F_4[-n,\, k+n,\, \mu+1,\, \nu+1,\, a^2,\, b^2]$$

$$\times \frac{g\Gamma\{\frac{1}{2}(\lambda+\tau+\kappa+2n-\rho)\}\Gamma\{\frac{1}{2}(\lambda+\tau+\kappa+2n+\rho)\}}{c^{\lambda+\kappa+2n+\tau}\,\Gamma(\tau+1)\Gamma(\kappa+2n+1)}$$

$$\times F_4\left[\tfrac{1}{2}(\lambda+\tau+\kappa+2n-\rho),\, \tfrac{1}{2}(\lambda+\kappa+\tau+2n+\rho), \tau+1, \kappa+2n+1,\, -\frac{1}{c^2},\, -\frac{g^2}{c^2}\right]$$

where $R(1+\kappa+\lambda+\tau) > |R(\rho)|$ and each of the numbers $R(c\pm i\pm ig)$ is positive

$$(3.2)\qquad \int_0^\infty J_\mu(at)J_\nu(bt)K_\rho(ct).\ t^{\kappa-\mu-\nu-\lambda}\,dt$$

$$= \frac{2^{\kappa-\mu-\nu-\lambda-1}a^\mu b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)}\sum_{n=0}^\infty \frac{(k+2n)\Gamma(k+n)}{n!}\,F_4[-n,\, k+n,\, \mu+1,\, \nu+1,\, a^2,\, b^2]$$

$$\times \frac{\Gamma\{\frac{1}{2}(k+2n-\lambda+\rho+1)\}\Gamma\{\frac{1}{2}(k+2n-\lambda-\rho+1)\}}{c^{k+2n-\lambda+1}\ \Gamma(k+2n+1)}$$

$$\times F[\tfrac{1}{2}(k+2n-\lambda+\rho+1),\, \tfrac{1}{2}(k+2n-\lambda-\rho+1),\, (k+2n+1),\, -\frac{1}{c^2}].$$

$$(3.3)\qquad \int_0^\infty J_\mu(at)J_\nu(bt)t^{\kappa-\mu-\nu+\lambda-1}\, e^{-p^2t^2}\,dt$$

$$= \frac{a^\mu.\, b^\nu.\, 2^{\kappa-\mu-\nu-1}}{\Gamma(\mu+1)\Gamma(\nu+1)}\sum_{n=0}^\infty \frac{(k+2n)\Gamma\{\lambda+\frac{1}{2}(k+n)\}}{(k+n)\, n!\, p^\lambda}$$

$$\times \frac{1}{(2p)^{k+n}} \times F_1\left(\frac{\lambda+k+n}{2},\, k+n+1, -\frac{1}{4p^2}\right),$$

where $\quad R\,(k+\lambda) > 0 \quad$ and $|\arg p| < \frac{\pi}{4}$.

(3.4)
$$\int_0^\infty J_\mu(at)J_\nu(bt)J_\tau(t)\, e^{-p^2t^2}\, t^{\kappa-\mu-\nu+\lambda-1}\, dt$$

$$= \frac{a^\mu\, b^\nu\,.\,2^{\kappa-\mu-\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(k+2n)}{(k+n)\;n!}\, F_4[-n,\, k+n,\, \mu+1,\, \nu+1,\, a^2,\, b^2]$$

$$\times \frac{\Gamma\left(\frac{\kappa+n+\lambda+\tau}{2}\right)}{2^{\kappa+n+\tau}\, p^{\kappa+n+\tau+\lambda}}$$

$$\times \frac{3F_3\left[\frac{\kappa+n+\tau+1}{2},\, \frac{\kappa+n+\tau+2}{2},\, \frac{\kappa+n+\tau+\lambda}{2} k+n+1,\, \tau+1,\; k+n+\tau+1,\, -\frac{1}{p^2}\right]}{\Gamma(\tau+1)},$$

where $\quad R(\tau+\lambda+1) > 0.$

(3.5)
$$\int_0^\infty J_\mu(at)J_\nu(bt)J_\tau(ct)\, t^{\kappa-\mu-\nu-\lambda}\, dt$$

$$= \frac{a^\mu.\, b^\nu.\, 2^{\kappa-\mu-\nu-\lambda}}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(k+2n)\Gamma(k+n)}{n!}\, F_4(-n,\, k+n,\, \mu+1,\, \nu+1,\, a^2,\, b^2)$$

$$\times \frac{c^\tau\, \Gamma\{\frac{1}{2}(k+n)+\frac{1}{2}\tau-\frac{1}{2}\lambda+\frac{1}{2}\}}{\Gamma(\tau+1)\Gamma\{\frac{1}{2}\lambda+\frac{1}{2}(k+n)+\frac{1}{2}-\frac{1}{2}\tau\}}$$

$$\times 2F_1\left(\frac{\tau+k+n-\lambda+1}{2}\,,\; \frac{\tau-\lambda-(k+n)+1}{2}\; ;\, \tau+1,\, c^2\right)$$

if $\;0<c<1 \qquad$ and

$$= \frac{2^{\kappa-\mu-\nu-\lambda}\, a^\mu.\, b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(k+2n)\Gamma(k+n)}{n!}\, F_4[-n,\, k+n,\, \mu+1,\, \nu+1,\, a^2,\, b^2]$$

$$\times \frac{\Gamma\{\frac{1}{2}\tau+\frac{1}{2}(k+n)-\frac{1}{2}\lambda+\frac{1}{2}\}\, 2F_1\left(\frac{\tau+k+n-\lambda+1}{2},\, k+n-\lambda-\tau+1,\, k+n+1,\, \frac{1}{c^2}\right)}{c^{k+n-\lambda+1}\;\Gamma(k+n+)\Gamma\{\frac{1}{2}\lambda+\frac{1}{2}\tau-\frac{1}{2}(k+n)+\frac{1}{2}\}}$$

if $\;c>1$

§4

Since

$$(4.1)\qquad k_{2n}(x)\Gamma(1+n) = W_{n,\frac{1}{2}}(2x)$$

we have after setting $m=\frac{1}{2}$ and $k=n$ in (3) and (1) of M.

$$(4.2)\qquad k_{2n}(x)k_{-2n}(x)$$

$$= \frac{-4x \sin n\pi}{n\pi}\int_0^\infty J_{2n}(\tfrac{1}{2}v^2)K_1(\sqrt{2}xv)\{J_1(\sqrt{2}xv)\cos n\pi$$

$$+Y_1(\sqrt{2}xv)\sin n\pi\}vdv$$

where $0<n<+\frac{1}{2}$ and

$$(4.3)\qquad k_{2n}\left(\frac{z^2 e^{\frac{1}{2}\pi i}}{2}\right)k_{2n}\left(\frac{z^2 e^{-\frac{1}{2}\pi i}}{2}\right)$$

$$= \frac{4z^2}{\{\Gamma(1+n)\}^2\Gamma(1-n)\Gamma(-n)}\int_0^\infty J_{-2n}(\tfrac{1}{2}v^2)k_1(zve^{\frac{1}{4}\pi i})k_1(zve^{-\frac{1}{4}\pi i})vdv$$

where $0>n>-\frac{1}{2}$

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The Theory of the Extensional Vibration of a bar excited by the Longitudinal Impact at the fixed end, the other end being free

BY

M Ghosh and S. C Dhar

The problem of the extensional vibrations of a bar, fixed at one end and free at the other and excited by the longitudinal impact of a hard load at the *free* or the *fixed* end was studied by Boussinesq * following St Venants' method of 'variation of integration constant.' In so doing, he divided the period of duration into a series of equal intervals, each being equal to the time taken by the longitudinal wave to travel from the struck end and back. The method consists in evaluating the unknown function by solving the Equation prometrico the obstinacy of which increases with intervals.

It is found that the duration of contact, when the load strikes at the *free* end, depends on the mass-ratio of the bar and the load But in the present case, it is equal to the period of the fundamental tone

In a recent paper,† one of the authors has extended the problem to the case of an elastic load, striking at the *free* end of the rod, and

* Boussinesq, Application des potentiels. Love's Elasticity, Third Edition, Art 282 441

† Ghosh, Bull Cal. Math Soc, **27**, *1 30* (1935).
Zeit. f. Angw. Math Mec., **14**, *71 78* (1934)
Ind. Phy Math. Jour., **3**, *73 79* (1932).

has made an improvement on the method of solving the problem. General expressions for any interval have been deduced.

In the present paper we study the case, when the elastic load strikes at the *fixed end* following the method adopted in the previous paper. It is found that, the duration of contact without being constant, changes with the elastic constant of the load.

The differential equation of the extensional vibration of the rod is

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial s^2}, \qquad \ldots \quad (1)$$

where, 'w' is the longitudinal displacement of the bar, 's' the distance measured from the free end, 't' the time measured from the beginning of the impact, 'c' the velocity of longitudinal wave propagation along the bar and is given by $c^2 = \mathrm{E}a/\rho$, 'E' being the Young's modulus of the material of the bar, 'ρ,' its mass density and a the cross section

The terminal condition at $s=0$ is $\dfrac{\partial w}{\partial s} = 0$ for all values of t, and at $s=l$, the terminal condition is the equation of motion of the striking body.

Since the pressure exerted by the load obeys Hooke's law, the equation of motion of the striking body of mass M is given by

$$\mathrm{P} = \mathrm{M}\left(\frac{\partial^2 z}{\partial t^2}\right) = \mathrm{E}a\left(\frac{\partial w}{\partial s}\right)_{s=l}$$

$$= -\epsilon\xi, \qquad \ldots \quad (2)$$

where, 'ϵ' is the elastic constant z, the displacement of the centre of gravity of the load, and is given by

$$z = w_{s=l} + \xi, \qquad \ldots \quad (3)$$

ξ being the compression of the load.

The solution of the equation (1) is of the form

$$w = \mathrm{F}(ct-s) + \psi(ct+s),$$

where F and ψ indicate two arbitrary functions.

From the terminal condition $\frac{\partial w}{\partial s} = 0$ at $s = 0$, the equation (3) can be written as

$$w = F(ct - s) + F(ct + s) \qquad , \qquad (4)$$

Now, from (2) and (4), we have,

$$\zeta = -\lambda\{F''(ct - l) - F''(ct + l)\} \qquad , \qquad (5)$$

where, $\lambda = Ea/c$

Hence the equation (2) becomes

$$F'''(\zeta) + \frac{1}{\lambda} F''(\zeta) + \frac{c}{Mc^2} F'(\zeta) = -\frac{2}{\lambda} F''(\zeta - 2l)$$

$$+ F'''(\zeta - 2l) + \frac{1}{\lambda} F''(\zeta - 2l) + \frac{c}{Mc^2} F'(\zeta - 2l) \qquad , \qquad (6)$$

where, ζ stands for $ct + l$

The integral of the eq. (6) is always of the form

$$F''(\zeta) = Ae^{q\zeta} + Be^{p\zeta} - \frac{2}{\lambda}\,\frac{1}{f(D)}\, F''(\zeta - 2l) + F''(\zeta - 2l) \qquad \dots \qquad (7)$$

where q and p are the roots of the equation

$$f(D) = D^2 + \frac{D}{\lambda} + \frac{c}{Mc^2} = 0,$$

and are given by

$$\{q, p\} = -\left\{\frac{c}{2Ea} \pm \sqrt{\frac{c^2}{4E^2a^2} - \frac{c}{Mc^2}}\right\} \qquad \dots \qquad (8)$$

When $3l > \zeta > l$,

$$F''(\zeta) = A_1 e^{q\zeta} + B_1 e^{p\zeta} \qquad \dots \qquad (9)$$

as
$$F'(\zeta-2l) - \frac{2}{\lambda}\,\frac{1}{f(D)}F''(\zeta-2l)$$

vanishes, for $F(\zeta-2l)$ is known in the interval $5l>\zeta>3l$

Now from the initial condition, *i e*, at $t=0$, $\xi=0$ and $\dot{z}=V$, eq. (9) becomes

$$F'(\zeta) = -\frac{V}{c\beta}\{e^{q(\zeta-l)}-e^{p(\zeta-l)}\}, \qquad \ldots \quad (10)$$

where
$$\beta = \lambda(q-p)$$

During the interval $5l>\zeta>3l$, we have from (10),

$$F''(\zeta-2l) = -\frac{V}{c\beta}\{qe^{q(\zeta-3l)}-pe^{p(\zeta-3l)}\} \qquad \ldots \quad (11)$$

Now from the condition of continuity of ζ and $\dot{z}$, at $ct=2l$, eq (7) with the help of eq., (11), becomes

$$F'(\zeta) = F'(\zeta) \text{ in eq } (10)+\frac{V}{c\beta^2}\left[\left\{2-\beta^2+2\beta q(\zeta-3l)\right\}e^{q(\zeta-3l)}\right.$$

$$\left.-\left\{2-\beta^2-2\beta p\,(\zeta-3l)\right\}e^{(\zeta-3l)}\right] \qquad \ldots \quad (12)$$

In a similar manner $F'(\zeta)$ for higher intervals can be calculated

$F(\zeta)$ can be easily obtained by integrating $F'(\zeta)$, and the constant of integration is to be found from the condition that there is no sudden change in $F(\zeta)$ at $s=l$ at the beginning of each interval.

From eqs. (2) and (5), the pressure exerted by the lead, is

$$P = -Ea[F'(\zeta - F'(\zeta-2l)] \qquad \ldots \quad (13)$$

So from eq (10) the pressure during the interval, $3l>\zeta>l$, is given by

$$P_1 = \frac{\rho Vc}{\beta}\,(e^{qct} - e^{pct}) \qquad \ldots \quad (14)$$

as $F(\zeta-2l)$ does not occur during this interval,

From eq (12), (13) becomes for the interval $5l > \zeta > 3l$

$$P_2 = P_1 \text{ in eq. (14)} - \frac{2\rho Vc}{\beta^2}\left[e^{q(ct-2l)}\{1+\beta q(ct-2l)\}\right.$$

$$\left. - e^{p(ct-2l)}\{1-\beta p(ct-2l)\}\right] \qquad \ldots \quad (15)$$

Now where ϵ is large compared with E, we have from (8)

$$\left.\begin{aligned} q &= -\frac{1}{ml}, \\ p &= -\frac{\epsilon}{Ec}, \end{aligned}\right\} \qquad \ldots \quad (16)$$

where $m = M/l\rho$, i.e., equal to the ratio of the mass of the load to that of the bar.

In the case of the rigid load, i.e., $\epsilon = \infty$, $F'(\zeta)$ in eqs (10) and (12) and the pressure P_1 and P_2 as given in eqs (14) and (15), become identical with those obtained by Boussinesq.

From eq (8), q and p become imaginary when $\frac{4Ea}{Ma^2} > \frac{\epsilon}{Ea}$, i.e., when the hammer is light and soft, and can be written as

$$\left.\begin{aligned} q &= \mu + i\nu, \\ p &= \mu - i\nu, \end{aligned}\right\} \qquad \ldots \quad (17)$$

where,

$$\left.\begin{aligned} \mu &= -\frac{\epsilon}{2Ea}, \\ \nu &= \sqrt{\frac{\epsilon}{Ma^2} - \frac{\epsilon^2}{4E^2a^2}} \end{aligned}\right\} \qquad \ldots \quad (18)$$

Hence we have,

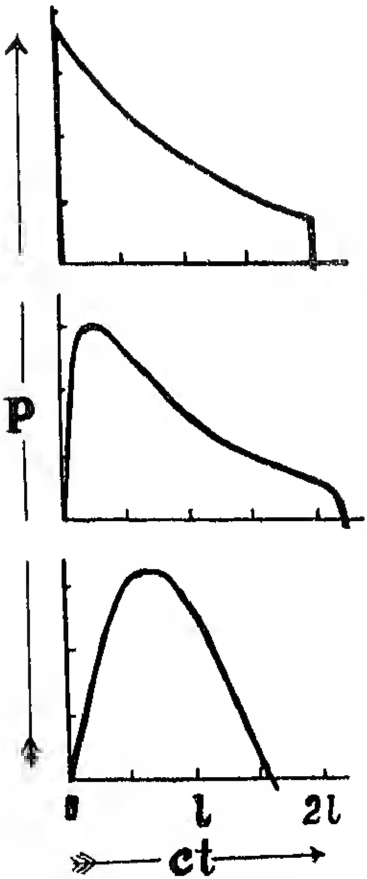
$$P_1 = \frac{\rho Vc}{\lambda\nu} e^{\mu ct} \sin \nu ct, \qquad \ldots \quad (19)$$

$$P_2 = P_1 \text{ in eq (20)} - \frac{\rho Vc}{\lambda^2\nu^2} e^{\mu(ct-2l)}\left[\sqrt{\mu^2+\nu^2}\,(ct-2l)\right.$$

$$\left. \times \sin\left(\nu ct - 2l - \tan^{-1}\frac{\mu}{\nu}\right) - \frac{\sin \nu(ct-2l)}{2\lambda\nu}\right] \qquad \ldots \quad (20)$$

Duration of contact

The duration of contact Φ is usually defined as the positive root of the pressure function equated to zero. The pressure terminates during the first interval, only when q and p are imaginary and is given by $\Phi = \frac{\pi}{pc}$. This is the lowest positive root of the eq. (20) equated to zero. In all other cases the pressure terminates at higher epochs depending on the elastic constant and the mass ratio of the load and the bar.



M=100 gms., E=10^{10} dy/cm^2,
P=8.5 gm., α=1, V=50 cm/sec.
l=10 cms.

Fig. 1. $\epsilon = \infty$, (hard load)

Fig. 2. $\epsilon/E = 1$, $\frac{\epsilon}{E\alpha} > \frac{4E\alpha}{Mc^2}$

Fig. 3. $\epsilon/E = 0.1$, $\epsilon/E\alpha < 4E\alpha/Mc^2$

It is easily seen from the above pressure equation that, in the case of a hard load, the duration of contact is constant and is equal to $2l/c$. At $t=0$ the pressure takes a sudden jump by ρVc (fig. 1). It then falls exponentially. In other cases (figs. 2, 3,) the pressure continuously increases, attains a maximum value and then gradually falls to zero. The time at which pressure falls to zero, depends mainly upon the elastic constant ϵ of the load. The adjoining curves (figs. 1 to 3) have been drawn by taking a concrete case of collision, to illustrate the three typical cases of impact. It is quite clear from the curves that in the case of elastic load the duration of contact is always greater than $2l/c$ except when the load is light and soft in which case the duration of contact is less than $2l/c$.

A NOTE ON THE VIBRATIONS OF A CIRCULAR RING

BY

S. GHOSH

The free vibrations of a rod which, in the unstressed state, forms a circular ring, have been discussed fully by several writers,* after the "rotatory inertia" terms have been neglected from the equations of motion. The retention of these terms, introduces no great complications in the problem and although the correction to be applied to the period of vibrations of rings of ordinary dimensions, is inappreciable for the graver modes, it is nevertheless as important as Pochhammer's correction to the period of vibrations of straight bars. The present note is intended for the examination of this correction to the period of vibration, due to rotatory inertia, and it is found that, as the number of wave lengths in the circumference increases, this correction increases in importance in the flexural vibrations, while it is small and remains practically stationary for torsional vibrations.

If c be the radius of the cross section of the ring and a that of its central line, the equations of motion are†

$$\left.\begin{aligned} \frac{\partial N}{\partial \theta} + T &= ma\frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial N'}{\partial \theta} &= ma\frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial T}{\partial \theta} - N &= ma\frac{\partial^2 w}{\partial t^2}, \end{aligned}\right\} \quad \ldots \quad (1)$$

* For full reference, see Love's Elasticity (4th ed.), pp. 451-54.

† Love, *loc. cit.*, p. 451.

and

$$\left.\begin{aligned}
\frac{\partial G}{\partial \theta}+H-N'a &= -\tfrac{1}{4}mc^2\,\frac{\partial^3 v}{\partial t^2\,\partial \theta}\,,\\
\frac{\partial G'}{\partial \theta}+Na &= \tfrac{1}{4}mc^2\,\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial \theta}+w\right),\\
\frac{\partial H}{\partial \theta}-G &= \tfrac{1}{2}mc^2 a\,\frac{\partial^2 \beta}{\partial t^2}\,,
\end{aligned}\right\}\qquad \dots \quad (2)$$

where

$$\left.\begin{aligned}
G &= \frac{E\pi c^4}{4a^2}\left(a\beta-\frac{\partial^2 v}{\partial \theta^2}\right),\\
G' &= \frac{E\pi c^4}{4a^2}\left(\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial w}{\partial \theta}\right),\\
H &= \frac{\mu\pi c^4}{2a^2}\left(\frac{\partial v}{\partial \theta}+a\,\frac{\partial \beta}{\partial \theta}\right),
\end{aligned}\right\}\qquad \dots \quad (3)$$

the symbols having their usual meanings.

The condition for inextensibility of the ring is

$$u = \frac{\partial w}{\partial \theta} \qquad \dots \quad (4)$$

Flexural vibrations in the plane of the ring.

Eliminating T and u from the first and third equations of (1) and equation (4), we have

$$\frac{\partial^2 N}{\partial \theta^2}+N = ma\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial \theta^2}-w\right) \qquad \dots \quad (5)$$

Also from the second equation of (2) and the second equation of (3) and equation (4), we have

$$N = -\frac{E\pi c^4}{4a^3}\left(\frac{\partial^4 w}{\partial \theta^4}+\frac{\partial^2 w}{\partial \theta^2}\right)+\frac{mc^2}{4a}\,\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial \theta^2}+w\right). \dots \quad (6)$$

Substituting from (6) in (5), we get

$$\frac{E\pi c^4}{4a^3}\left(\frac{\partial^6 w}{\partial\theta^6}+2\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial^2 w}{\partial\theta^2}\right)$$

$$= ma\frac{\partial^2}{\partial t^2}\left(w-\frac{\partial^2 w}{\partial\theta^2}\right)+\frac{mc^2}{4a}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 w}{\partial\theta^4}+2\frac{\partial^2 w}{\partial\theta^2}+w\right) \qquad (7)$$

Assuming that

$$w = W e^{i(n\theta+pt)}, \qquad \ldots \quad (8)$$

where n is an integer, we have

$$\frac{E\pi c^4}{4ma^4}n^2(n^2-1)^2 = p^2(n^2+1)\left[1+\frac{1}{4}\frac{(n^2-1)^2}{n^2+1}\frac{c^2}{a^2}\right],$$

which gives

$$p^2=\frac{E\pi c^4}{4ma^4}\frac{n^2(n^2-1)^2}{n^2+1}\left[1+\frac{1}{4}\frac{(n^2-1)^2}{n^2+1}\frac{c^2}{a^2}\right]^{-1} \qquad (9)$$

Neglecting c^2/a^2, we get Hoppe's frequency equation

$$p^2 = \frac{E\pi c^4}{4ma^4}\frac{n^2(n^2-1)^2}{n^2+1}. \qquad \ldots \quad (10)$$

The effect of rotatory inertia is therefore to increase the period $(2\pi/p)$ of vibration of the ring in the ratio

$$1 : 1+\frac{1}{8}\frac{(n^2-1)^2}{n^2+1}\cdot\frac{c^2}{a^2}$$

Now as the fraction $(n^2-1)^2/(n^2+1)$ continually increases with n, it follows that the effect becomes more and more marked as n increases.

The following table gives the increment per cent in the period of vibration of a ring for which $c/a = 1/10$

n	2	3	4	5	6	7	8	9	10
Increment per cent	·2	·8	1·6	2·8	4·1	5·8	7·3	9·7	12·1

For the graver modes, the correction is nearly insensible, but it becomes appreciable as n increases.

Flexural vibrations at right angles to the plane of the ring.

Eliminating N′ between the second equation of (1) and the first equation of (2) and then substituting for G and H from (3), we get

$$\frac{E\pi c^4}{4ma^4}\left(a\frac{\partial^2\beta}{\partial\theta^2}-\frac{\partial^4 v}{\partial\theta^4}\right)+\frac{\mu\pi c^4}{2ma^4}\left(\frac{\partial^2 v}{\partial\theta^2}+a\frac{\partial^2\beta}{\partial\theta^2}\right)$$

$$=\frac{\partial^2}{\partial t^2}\left(v-\frac{1}{4}\frac{c^2}{a^2}\frac{\partial^2 v}{\partial\theta^2}\right) \quad \ldots \quad (11)$$

Substituting from (3) in the last equation of (2) we have

$$-\frac{E\pi c^4}{4ma^4}\left(a\beta-\frac{\partial^2 v}{\partial\theta^2}\right)+\frac{\mu\pi c^4}{2ma^4}\left(\frac{\partial^4 v}{\partial\theta^4}+a\frac{\partial^2\beta}{\partial\theta^2}\right)$$

$$=\frac{1}{2}\frac{c^2}{a^2}\,a\frac{\partial^2\beta}{\partial t^2} \quad \ldots \quad (12)$$

Assuming that

$$v=Ve^{\iota(n\theta+pt)},\qquad a\beta=Be^{\iota(n\theta+pt)},$$

where n is an integer, we get

$$\left.\begin{aligned}&\left(1+\frac{2\mu}{E}\right)n^2B+\left[n^4+\frac{2\mu}{E}n^2-\left(1+\frac{1}{4}\frac{n^2c^2}{a^2}\right)\left(\frac{4ma^4p^2}{E\pi c^4}\right)\right]V=0\\&\left[1+\frac{2\mu}{E}n^2-\frac{1}{2}\frac{c^2}{a^2}\left(\frac{4ma^4p^2}{E\pi c^4}\right)\right]B+\left(1+\frac{2\mu}{E}\right)n^2V=0\end{aligned}\right\}\quad(13)$$

Eliminating B : V between the equations (13), we get the frequency equation as

$$\left(1+\frac{2\mu}{E}\right)^2n^4=\left[n^4+\frac{2\mu}{E}n^2-\left(1+\frac{1}{4}\frac{n^2c^2}{a^2}\right)\left(\frac{4ma^4p^2}{E\pi c^4}\right)\right]$$

$$\times\left[1+\frac{2\mu}{E}n^2-\frac{1}{2}\frac{c^2}{a^2}\left(\frac{4ma^4p^2}{E\pi c^4}\right)\right] \quad \ldots \quad (14)$$

Simplifying the equation and using the relation $E/2\mu=1+\sigma$, we get

$$n^2(n^2-1)^2$$

$$=(n^2+1+\sigma)\left(1+\frac{1}{4}\frac{n^2c^2}{a^2}\right)\left[1-\frac{1}{2}\frac{1+\sigma}{n^2+1+\sigma}\frac{c^2}{a^2}\left(\frac{4ma^4p^2}{E\pi c^4}\right)\right]\left(\frac{4ma^4p^2}{E\pi c^4}\right)$$

$$+\frac{1}{2}n^2(n^2+1+\sigma n^2)\frac{c^2}{a^2}\left(\frac{4ma^4p^2}{E\pi c^4}\right) \quad \ldots \quad (15)$$

If we neglect c^2/a^2, we get Michell's frequency equation

$$\frac{4ma^4p^2}{E\pi c^4} = \frac{n^2(n^2-1)^2}{n^2+1+\sigma} \qquad \ldots \quad (16)$$

for flexural vibrations at right angles to the plane of the ring.

If in the terms containing c^2/a^2 as a factor, we substitute for p^2 its approximate value, we get

$$(n^2+1+\sigma)\left(\frac{4ma^4p^2}{E\pi c^4}\right) = n^2(n^2-1)^2\left[1 - \tfrac{1}{4}n^2\frac{c^2}{a^2}\right.$$

$$\left. + \tfrac{1}{4}\frac{(1+\sigma)n^2(n^2-1)^2}{(n^2+1+\sigma)^2}\frac{c^2}{a^2} - \tfrac{1}{4}\frac{n^2(n^2+1+\sigma n^2)}{n^2+1+\sigma}\frac{c^2}{a^2}\right]$$

$$= n^2(n^2-1)^2\left[1 - \tfrac{1}{4}\frac{n^2c^2}{a^2} - \tfrac{1}{4}\frac{(2+\sigma)^2n^4}{(n^2+1+\sigma)^2}\cdot\frac{c^2}{a^2}\right]. \quad (17)$$

The effect of rotatory inertia is therefore to increase the period of vibration of the ring in the ratio

$$1 : 1+\left[\tfrac{1}{8}n^2 + \tfrac{1}{8}\frac{(2+\sigma)^2n^4}{(n^2+1+\sigma)^2}\right]\frac{c^2}{a^2},$$

so that the correction to the period becomes greater and greater as n increases.

The following table gives the increment per cent. in the period of vibration of a ring for which $c/a = 1/10$ and $\sigma = \cdot 3$.

n	2	3	4	5	6	7	8	9	10
Increment per cent.	1·3	2·1	3·1	4·3	5·7	7·3	9·3	11·4	13·8

For graver modes, this correction is small, but the percentage of increment is much greater than in the case of flexural vibrations in the plane of the ring.

Torsional vibrations.

Returning to the equation (15), let us study the short period vibrations. If we write the equation in the form

$$\left[(n^2+1+\sigma)\left(1+\tfrac{1}{4}\frac{n^2c^2}{a^2}\right) + \tfrac{1}{4}n^2(n^2+1+\sigma n^2)\frac{c^2}{a^2}\right]\left(\frac{E\pi c^4}{4ma^4p^2}\right)$$

$$= \tfrac{1}{2}(1+\sigma)\,\frac{c^2}{a^2}\left(1+\tfrac{1}{2}\,\frac{n^2c^2}{a^2}\right) + n^2(n^2-1)^2\left(\frac{E\pi c^4}{4ma^4p^2}\right)^2. \quad (18)$$

and neglect squares and products of $(E\pi c^4)/(4ma^4p^2)$ and c^2/a^2, we get as a first approximation, Basset's frequency equation

$$(n^2+1+\sigma)\left(\frac{E\pi c^4}{4ma^4p^2}\right) = \tfrac{1}{2}(1+\sigma)\frac{c^2}{a^2}. \quad \ldots \quad (19)$$

for torsional vibrations of the ring

For a second approximation, we substitute this value of p^2 in the terms neglected in the first approximation, and we get

$$(n^2+1+\sigma)\left(\frac{E\pi c^4}{4ma^4p^2}\right)$$

$$= \tfrac{1}{2}(1+\sigma)\,\frac{c^2}{a^2}\left[1+\tfrac{1}{2}.\,\frac{(1+\sigma)n^2(n^2-1)^2}{(n^2+1+\sigma)^2}.\,\frac{c^2}{a^2} - \tfrac{1}{2}\,\frac{n^2(n^2+1+\sigma n^2)}{n^2+1+\sigma}.\,\frac{c^2}{a^2}\right].$$

$$= \tfrac{1}{2}(1+\sigma)\frac{c^2}{a^2}\left[1 - \tfrac{1}{2}\,\frac{(2+\sigma)^2n^4}{(n^2+1+\sigma)^2}.\,\frac{c^2}{a^2}\right] \quad \ldots \quad (20)$$

The effect of rotatory inertia is thus to decrease the period of vibration in the ratio

$$1 : 1 - \tfrac{1}{4}\,\frac{(2+\sigma)^2n^4}{(n^2+1+\sigma)^2}.\,\frac{c^2}{a^2}.$$

Since $n^4/(n^2+1+\sigma)^2$ is less than 1, the correction per cent. is less than

$$25(2+\sigma)^2\,\frac{c^2}{a^2}$$

The following table gives the correction per cent. in the period of vibration of a ring for which $c/a = 1/10$ and $\sigma = 3$

n	1	2	3	4	5	6	7	8	9	10
Diminution per cent	25	·8	1	1·1	1·2	1·2	1·2	1·3	1·3	1·3

On the Irreducible invariants and covariants system of two quaternary Quadrics and two Linear Complexes

BY

N Chatterjee and P N Dasgupta

(*Communicated by S Mukhopadhyaya*)

Introduction

By the use of complex symbols Weitzenböck* has considered the invariants and covariants system of a quaternary quadric associated with two linear complexes From the view-point of a Prepared System Turnbull has discussed the concomitants of a system of n Linear complexes † The complete system which includes linear complexes and mixed concomitants of a quadric with two linear complexes has been discussed by one of us elsewhere ‡ The present paper deals with the invariants and covariants system of two quaternary quadrics associated with two linear complexes.

Notation §

1. The symbols x, u, p denote homogeneous co-ordinates such that

$$x = x_1, x_2, x_3, x_4 \qquad \text{(point co-ordinates)},$$

$$u = u_1, u_2, u_3, u_4 \qquad \text{(plane co ordinates)},$$

$$p = p_{12}, p_{23}, p_{13}, p_{14}, p_{24}, p_{34} \quad \text{(line co ordinates)},$$

$$= \left\| \begin{matrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \end{matrix} \right\|,$$

* "Zum System eines linearen Komplexes und einer Fläche zweiter Ordnung," Journal für Math., **137** (1910), 65-82.

† Das Gupta, Proc. Lond Math. Soc , Ser **2**, **31** Part 7.

‡ "On the invariant theory of mixed quaternary forms," Proc Lond. Math. Soc., Ser. 2, **25**, Parts 4 and 5

§ This notation and its applications are more fully explained in a paper by Turnbull, Proc. Lond. Math. Soc., 2, **25** (1920), 303-327.

where y is cogredient to x, and

$$p_{ik} = x_i y_k - x_k y_i.$$

Also $\qquad a_x = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$

The quadric $\quad \Sigma a_{ij} x_i x_j \quad$ is denoted by

$$f = a_x^2 = a'^2_x = a''^2_x = \ldots,$$

where a, a', a'' …… are equivalent symbols, the other quadric $\Sigma b_{ij} x_i x_j$ has corresponding implications

In what follows $\quad A = a_1 a_2\,, \quad \alpha = a a' a''\,,$
$\quad B = b_1 b_2\,; \quad \beta = b b' b''\,,$

such that $\qquad a_\alpha = (a a' a'' a''') = \Sigma(a_1 a'_2 a''_3 a_4''')$,

a determinant of the fourth order and so on

The linear complexes are denoted by (Kp), (Lp), (Mp), … . The co efficient K is represented by two symbols k_1, k_2 both cogredient with a, so that $K_{lm} = k_{1l} k_{2m} - k_{2l} k_{1m}$ Similarly, $A = a$, a', etc, where elements a, a', b, b', k_1, k_2, .. are of currency one, contragredient to A, K, … are A, K, … which are each of currency two, while contragredient to a, b, … are the symbols α, β, … which are each of currency three.

Let

$$(KL) = K_{12}L_{34} + K_{13}L_{14} + K_{31}L_{24} + K_{14}L_{23} + K_{24}L_{31} + K_{34}L_{12}.$$

Similarly, let

$$(AB) = A_{12}B_{34} + A_{23}B_{14} + \ldots + A_{34}B_{12},$$

where $\qquad A_{12} = \begin{vmatrix} a_1 & a'_1 \\ a_2 & a'_2 \end{vmatrix}, \quad B_{34} = \text{etc.}$

Besides the usual bracket factors certain compound factors are useful e g.,

$(bKAx)$ is a compound factor such that

$$(bKAx) = (bKa_1)(a_2 x) - (bKa_2)(a_1 x)$$

$$= (bK\dot{a}_1)(\dot{a}_2 x), \quad \text{where dot denotes determinantal permutation}$$

$$= \Omega_a (bKa_1)(a_2 x)\ *$$

Also

$$(bKALMx) = \Omega_a (bKa_1)(a_2 LMx)$$

$$= \Omega_a \Omega_m (bKa_1)(a_2 Lm_1)(m_2 x)$$

$$= \Omega_{am} (bKa_1)(a_2 Lm_1)(m_2 x),$$

* Turnbull, Proc Royal Soc. Edin., **46**, (1926), *210 222*.

where Ω_{am} denotes a sum of four terms obtained by permuting a_1, a_2 and m_1, m_2 independently with proper signs.

Where the end elements are of currency two, the bracket factor is such that

$$(\mathrm{AKLMNQ}) = (a_1\mathrm{KLMNQ}a_2) = \Omega_a(a_1\mathrm{KLMNQ}a_2)$$

2. For the purpose of discussing properties of the bracket factors the following identities are fundamental:*

$$(u\mathrm{KA}x)+(u\mathrm{AK}x) = -(\mathrm{AK})u_x,$$

or, more generally,

$$(u\mathrm{KALM}x) + (u\mathrm{AKLM}x) = -(\mathrm{AK})(u\mathrm{LM}x), \quad .. \quad (1)$$

$$(u\mathrm{KLMN}x) = -(x\mathrm{NMLK}u), \quad .. \quad (2)$$

$$(u\mathrm{KLLN}x) = -\tfrac{1}{2}(\mathrm{LL})(u\mathrm{KN}x), \quad ... \quad (3)$$

$$(x\mathrm{PQR}x) = -(x\mathrm{QPR}x) = (x\mathrm{QRP}x) = \text{etc} \quad .. \quad (4)$$

$$(\mathrm{AKLN}) = (a_1\mathrm{KLN}a_2)$$
$$= -(\mathrm{KL})(\mathrm{NA})-(\mathrm{LN})(\mathrm{KA})+(\mathrm{KN})(\mathrm{LA}), \quad ... \quad (5)$$

where $\quad \mathrm{A} = a_1 a_2.$

These identities have been proved elsewhere †

The proofs are reproduced here

Proof of identity (1).

$$(u\mathrm{KALM}x) = \Omega_a(u\mathrm{K}a_1)(a_2\mathrm{LM}x),$$
$$= (a_2\mathrm{K}a_1)(u\mathrm{LM}x)+\Omega_k(uk_1a_2a_1)(k_2\mathrm{LM}x)$$
$$= -(\mathrm{AK})(u\mathrm{LM}x)-(u\mathrm{AKLM}x),$$

where $\quad \mathrm{A} = a_1a_2, \quad \mathrm{K}=k_1k_2$

Proof of identity (2).

$$(u\mathrm{KLMN}x) = \Omega_{l,n}(u\mathrm{K}l_1)(l_2\mathrm{M}n_1)(n_2x),$$
$$= \Omega_{l,n}(xn_2)(n_1\mathrm{M}l_1)(l_2\mathrm{K}u),$$
$$= -(x\mathrm{NMLK}u)$$

Identities (3) and (4) follow immediately from (1) for a factor of of the type $(x\mathrm{R}x) = -(x\mathrm{R}x) = 0.$

Identity (5) is proved in a similar way.

* Turnbull, Theory of Determinants (1928), 210 212.

† Turnbull, Proc Lond. Math. Soc., 2, 25 (1926), 303 327.

Prepared System

3 To render all convolutions of convenient forms explicit, a system of symbolic types has been evolved to which the name Prepared System* has been given. By the fundamental theorem, all concomitants are capable of being evolved out of bracketed factors of the simple type (u_x), (AK), $(a\mathrm{K}u)$, (apb) or of bracket factors of the compound type $(b\mathrm{KA}\alpha)$, $(a\mathrm{KL}\alpha)$, etc. Since we are considering concomitants of quadrics, each of the symbols a, A, α or b, B, β in a concomitant form must occur twice These are quadric symbols It is not necessary to bracket two quadric symbols when they occur in a compound bracket factor to pair the identical symbols in another compound bracket factor for they may appear separately in other bracket factors simple or compound. It should be noticed that whenever a quadric symbol, e g, a occurs in a concomitant, the variants of that symbol, e g., a_1 or a_2 or a_3 should be taken all equivalent to a. The complete Prepared System for two linear complexes $(\mathrm{K}p)$, $(\mathrm{L}p)$ and two quadrics a_x^2 and b_x^2 is given in the following Table A It is necessary to note that no forms could contain a_α or b_β for as soon as four equivalent symbols are found together, similar combination can be made by collecting together the symbols a and α from the remaining portion of the concomitant providing for a_α^2 which itself is an invariant. Similar considerations preclude the inclusion of a and A symbols of the same quadric of currency one and two for they might come together to give a quadric symbol α of currency three. This explains how factors of the type $(a ... \alpha)$, $(a .. \beta)$ are admissible while factors of the type $(a\mathrm{KA}\beta)$, $(a\mathrm{LA})$ are not admissible. Where there is a bracket factor involving one linear complex, e.g , K, the corresponding form with the complex L is not mentioned with a view to avoid repetition, for in this paper we confine ourselves to the consideration of representative forms

We use the numbers 1, 2, 3 to denote the quadric symbols a, A, α respectively while 1′, 2′, 3′ stand for the corresponding symbols b, B, β for the second quadric. The forms are listed according to convolution of quadric symbols.

* Turnbull, Proc. Lond. Math Soc., 2, 2′ (1922) and 2, 25 (1926), *303 327*, also Theory of Determinants (1928), *210 212*

TABLE A.

The Prepared forms for the concomitants, in general, for two quadrics and two linear complexes.

0	(KL), (KK'), Kp, u_x
1	a_x, (aKu), $(aKLx)$.
1′	b_x, (bKu), $(bKLx)$.
2	(Ap), (AK), $(uKAx)$, $(xKALx)$, $(uKALu)$
2′	(Bp), (BK), $(uKBx)$, $(xKBLx)$, $(uKBLu)$
3	u_α, $(uKL\alpha)$, (αKx), (αKLu).
3′	u_β, $(uKL\beta)$, (βKx), (βKLu)
(11′)	(apb), (aKb), $(aKpLb)$
(12′).	(aBu), $(aBKx)$, $(aKBLu)$
(1′2)	(bAu), $(bAKx)$, $(bKALu)$
(13)	$(apK\alpha)$, $(aKL\alpha)$
(1′3′)	$(bpK\beta)$, $(bKL\beta)$.
(13′):	a_β, $(aKL\beta)$.
(1′3).	b_α, $(bKL\alpha)$.
(22′).	(AB), $(uABx)$, $(uABKu)$, $(xAKBx)$, $(AKLB)$, $(uKABLx)$.
(23′):	$(xA\beta)$, $(uAK\beta)$, $(aKAL\beta)$.
(2′3):	$(xB\alpha)$, $(uBK\alpha)$, $(xKBL\alpha)$
(33′):	$(\alpha K\beta)$, $(\alpha p\beta)$, $(\alpha KpL\beta)$
(12′1):	$(aKBLa)$, $(apBLa)$.
(1′21′)	$(bKALb)$, $(bpALb)$
(12′3)	$(aKB\alpha)$, $(apB\alpha)$.
(123′):	$(bKA\beta)$, $(bpA\beta)$
(32′3)	$(\alpha KBL\alpha)$, $(\alpha pBL\alpha)$.
(3′23′)	$(\beta KAL\beta)$, $(\beta pAL\beta)$.

4. As in the present paper we confine ourselves to the consideration of invariant and covariant forms only, it will be seen that the Prepared forms, we shall have to deal with, will be limited. In subsequent work, we denote, by the introduction of underlines, forms which involve both complex symbols K, L. Thus, for instance, if by (1) we denote a_x, then $(\underline{1})$ will denote $(aKLx)$. The following Table B will indicate clearly the notation used in the subsequent work in connection with bracketed numeral factors.

TABLE B.

The prepared forms for invariants and covariants for two quadrics and two linear complexes.

$(1)=a_x$,	$(\underline{1})=(aKLx)$,	$(1')=b_x$,	$(\underline{1'})=(bKLx)$
$(2)=(AK)$,	$(\underline{2})=(xKALx)$,	$(2')=(BK)$,	$(\underline{2'})=(xKBLx)$.
$(3)=(\alpha Lx)$	,	$(3')=(\beta Lx)$.	
$(11')=(aKb)$			
$(12')=(aKBx)$	,	$(1'2)=(bKAx)$.	
$(13')=a_\beta$,	$(\underline{13'})=(aKL\beta)$;	$(1'3)=b_\alpha$,	$(\underline{1'3})=(bKL\alpha)$.
	$(\underline{13})=(aKL\alpha)$;		$(\underline{1'3'})=(bKL\beta)$
$(22')=(AB)$,	$(\underline{22'})=(xAKBx)$.		
$(23')=(xA\beta)$,	$(\underline{23'})=(xKAL\beta)$,	$(2'3)=(xB\alpha)$,	$(\underline{2'3})=(xKBL\alpha)$
$(33')=(\alpha K\beta)$.			
$(12'1)=(aKBLa)$	,	$(1'21')=(bKALb)$.	
$(12'3)=(aKB\alpha)$	;	$(1'23')=(bKA\beta)$.	
$(32'3)=(\alpha KBL\alpha)$	,	$(3'23')=(\beta KAL\beta)$.	

From this Table B, a complete system of invariants and covariants, including all irreducible forms and some redundancies, is at once written down. We now proceed to a discussion of the formulæ which enable us to reduce some of the concomitants.

Reducibility.

5 THEOREM *Identity formulae by expansion can always be attempted from the product of two bracket factors when any two of the four end elements are distinct and their currencies add to two or six, it being provided that an end element of currency one can be attached to the adjoining element of grade two to form a compound single element of currency three for the purpose of the proposed expansion*

The first part of the above theorem has been noticed elsewhere,* e.g., it has been shown that where a and x are both of currency three, the form

$$(u\mathrm{KL}\,ax\,\mathrm{MNQ}\lambda) = (u\mathrm{KL}a)(x\mathrm{MNQ}\lambda) - (u\mathrm{KL}x)(a\mathrm{MNQ}\lambda),$$

and again,

$$(u\mathrm{KL}\,ax\,\mathrm{MNQ}\lambda) = -(a\mathrm{L}x)(u\mathrm{KMNQ}\lambda) + (a\mathrm{K}x)(u\mathrm{LMNQ}\lambda)$$
$$+(u\,ax\,\mathrm{KLMNQ}\lambda)$$

Hence we get the identity

$$(u\mathrm{KL}a)(x\mathrm{MNQ}\lambda) - (u\mathrm{KL}x)(a\mathrm{MNQ}\lambda) = -(a\mathrm{L}x)(u\mathrm{KMNQ}\lambda)$$
$$+(a\mathrm{K}x)(u\mathrm{LMNQ}\lambda) + (u\,ax\,\mathrm{KLMNQ}\lambda) \quad \ldots \quad (6)$$

Again, if the paired elements are of currency one each, we have

$$(u\mathrm{KLM}\,ba\,\mathrm{NPQ}v) = (u\mathrm{KLM}b)(a\mathrm{NPQ}v) - (u\mathrm{KLM}a)(b\mathrm{NPQ}v),$$

and again since

$$(u\mathrm{KLM}.ba.\mathrm{NPQ}v) = -(b\mathrm{M}a)(u\mathrm{KLNPQ}v) + (b\mathrm{L}a)(u\mathrm{KMNPQ}v)$$
$$+(u\mathrm{K}.ba.\mathrm{LMNPQ}b),$$

we have an identity

$$(u\mathrm{KLM}b)(a\mathrm{NPQ}v) - (u\mathrm{KLM}a)(b\mathrm{NPQ}v) = -(b\mathrm{M}a)(u\mathrm{KLNPQ}v)$$
$$+(b\mathrm{L}a)(u\mathrm{KMNPQ}v) + (u\mathrm{K}.ba.\mathrm{LMNPQ}b). \quad \ldots \quad (7)$$

We now proceed to show *that $a\mathrm{L}$ could be paired to behave as an element of currency three.*

By permutation $(a\,\mathrm{MNP}a)(x\mathrm{QRS}x)$

$$= \Omega_{n_1 n_2} \Omega_{q_1 q_2} (a\mathrm{M}n_1)(n_2\mathrm{P}a)(xq_1)(q_2\mathrm{RS}x)$$

$$= \Omega_{n_1 n_2} \Omega_{q_1 q_2} [(a\mathrm{M}q_1)\,n_2\mathrm{P}a)(xn_1)(q_2\mathrm{RS}x)$$

* Das Gupta and Turnbull "On the complete system of Linear Complexes," Proc Edin Math. Soc., 1920, 01 70.

$$+ \ (q_1 M n_1)(n_2 P a)(x a)(q_2 R S x) + (a q_1 m_2 n_1)(n_2 P a)(x m_1)(q_2 R S x)]$$

$$= (aMQRSx)(xNPa) + a_x(aPNMQRSx) + (aq_1 m_2 NPa)(xm_1)(q_2 RSx)$$

$$= (aMQRSx)(xNPa) + a_x(aPNMQRSx) + \underset{q_1 q_2}{\Omega}(xMaq_1 NPa)(q_2 RSx)$$

$$= (aMQRSx)(xNPa) + a_x(aPNMQRSx) - (aNQRSx)(xMPa)$$

$$+ (xMNa)(aPQRSx) - (xMNQRSx)(aPa).$$

Hence

$$(aMNPa)(xQRSx) = (aMQRSx)(xNPa) - a_x(aPMNQRSx)$$

$$-(aNQRSx)(xMPa) - (xNMa)(aPQRSx) \qquad \ldots \quad (8)$$

Again, using compound notation we get the same result symbolically in a few steps, for

$$(aMN \ aP/x \ QRSx) = (aMNPa)(xQRSx) - (aMNx)(aPQRSx)$$

The left hand member is again equal to

$$-(aPNx)(aMQRSx) + (aPMx)(aNQRSx) + (aPa)(xMNQRSx)$$

$$-a_x(aPMNQRSx).$$

Hence we have

$$(aMNPa)(xQRSx) = -(xNMa)(aPQRSx) + (xNPa)(aMQRSx)$$

$$-(xMPa)(aNQRSx) - a_x(aPMNQRSx) \quad \ldots \quad (9)$$

It is readily seen that the identities (8) and (9) are one and the same. This establishes the theorem.

6. $(aKBa)(xA\beta)$ *is reducible or symbolically* $(12'3)(23')=0.$

Proof.—

From the two-fold expansion of $(aKB.\ a/x\ \ A\beta)$ in the manner indicated in § 5, we have

$$(aKBa)(xA\beta) - (aKBx)(aA\beta) = -(aBx)(aKA\beta) + (aKx)(aBA\beta)$$

$$+ a_a(xKBA\beta) - a_x(aKBA\beta)$$

which shews that $(aKBa)(xA\beta)$ is reducible

7 $(aKBa)(xKAL\beta)$ *is reducible or symbolically* $(12'3)(23')=0.$

Proof :—

From the two fold expansion of $(a\mathrm{KB}\ a/x\ \mathrm{KAL}\beta)$ in the manner indicated in § 5, we have

$$(a\mathrm{KB}a)(x\mathrm{KAL}\beta) - (a\mathrm{KB}x)(a\mathrm{KAL}\beta)$$
$$= -(a\mathrm{B}x)(a\mathrm{KKAL}\beta) + (a\mathrm{K}x)(a\mathrm{BKAL}\beta) + a_a(x\mathrm{KBKAL}\beta) - a_x(a\mathrm{KBKAL}\beta)$$

from which it is evident that $(a\mathrm{KB}a)(x\mathrm{KAL}\beta)$ is reducible

8 $\quad a_\beta(x\mathrm{KBL}a) = (a\mathrm{L}\beta)(x\mathrm{KB}a) + (x\mathrm{K}\beta)(a\mathrm{BL}a)$

or symbolically $\quad (13')(2'\underline{3}) = (33')(12') + 3'(12'3).$

Proof :—

From the expansion of $(x\mathrm{KBL}\ a/\beta.\ a)$ in the two-fold manner we have

$$(x\mathrm{KBL}a)(\beta a) - (x\mathrm{KBL}\beta)(aa)$$

$$= -(a\mathrm{L}\beta)(x\mathrm{KB}a) + (a\mathrm{B}\beta)(x\mathrm{KL}a) - (x\mathrm{K}a)(\beta\mathrm{BL}a) + (x\mathrm{K}\beta)(a\mathrm{BL}a)$$

from which we get

$$a_\beta\ (x\mathrm{KBL}a) = (a\mathrm{L}\beta)(x\mathrm{KB}a) + (x\mathrm{K}\beta)(a\mathrm{BL}a)$$

9 $\quad (a\mathrm{KL}x)(b\mathrm{LA}\beta) = (a\mathrm{L}b)(x\mathrm{KAL}\beta) = (b\mathrm{LA}x)(a\mathrm{KL}\beta)$

or symbolically $\quad \underline{1}\ (1'23') = (11')(\underline{23'}) = (1'2)(\underline{13'})$

Proof :—

From the expansions of $(x\mathrm{KAL}.\ \beta/\mathrm{L}a.\ b)$ according to §5 we have

$$(x\mathrm{KAL}\beta)(a\mathrm{L}b) - (x\mathrm{KALL}a)(\beta b)$$
$$= -(\beta\mathrm{LL}a)(x\mathrm{KA}b) + (\beta\mathrm{AL}a)(x\mathrm{KL}b) + (x\mathrm{K}\beta)(a\mathrm{LAL}b) - (x\mathrm{KL}a)(\beta\mathrm{AL}b)$$

from which we obtain

$$(x\mathrm{KAL}\beta)(a\mathrm{L}b) = (a\mathrm{KL}x)(b\mathrm{LA}\beta).$$

Again, from $(b\mathrm{LA}\ x/\beta\ \mathrm{LK}a)$, we get

$$(b\mathrm{LA}x)(\beta\mathrm{LK}a) - (b\mathrm{LA}\beta)(x\mathrm{LK}a)$$

$$= -(x\mathrm{A}\beta)(b\mathrm{LLK}a) + (x\mathrm{L}\beta)(b\mathrm{ALK}a) + b_x(\beta\mathrm{LALK}a) - b_\beta(x\mathrm{LALK}a)$$

which yields the identity

$$(b\mathrm{LA}x)(a\mathrm{KL}\beta) = (a\mathrm{KL}x)(b\mathrm{LA}\beta)$$

10 $\qquad (a\mathrm{KB}a)(a\mathrm{L}b) = b_a(a\mathrm{KBL}a)$

or symbolically $\quad (11')(12'3) = (1'3)(12'1)$

Proof —

From the expansions of $(aBK\ a/b\ La)$, according to §5 we have

$$(aBKa)(bLa) - (aBKb)(aLa)$$
$$= -(aKb)(aBLa) + (aBb)(aKLa) + (aa)(bBKLa) - (ab)(aBKLa)$$

which yields $$(aKBa)(aLb) = b_a(aKBLa)$$

11 $$(aKBLa)(b_a) = (aLb)(aKBa) + (aKB)(aLBa)$$

or *symbolically* $$(1'3)(12'1) \longrightarrow (11')(12'3)$$

Proof —

From $(aKBL.a/b\ a)$ by the two-fold expansion of §5 we get

$$(aKBLa)(b_a) - (aKBLb\ a_a$$
$$= -(aLb)(aKBa) + aBb)(aKLa) + (aKa)(bBLa) - (aKb)(aBLa)$$

from which we are led to the one-way identity

$$(1'3)(12'1) \longrightarrow (11')(12'3).$$

12 "The $a-a$ theorem"

$$(a\ PQa)(\sigma\rho)(aR\sigma) = (aPQa)(a\sigma)(aR\rho).$$

This has been noticed elsewhere.* This theorem enables us to effect numerous reductions of which the following are typical —

(*i*) $$(aKLa)(aKLv)(aKx)$$
$$= (aKLa)(a/KLv)(aK/x)$$
$$= (aKLa)(a_x)(aKKLv)$$
$$= 0$$

(*ii*) $$a_x(aKx)(aLb)(aK\beta)(bKL\beta)$$
$$= a_x(aKx)(b/La)(\beta K/a)(bKL\beta)$$
$$= a_x(aKv)(b_a)(\beta KLa)(bKL\beta)$$

13. The reduction and equivalence formulae, obtained by pairing the prepared forms two and two, are exhibited in Table C annexed. In the Table C blanks indicate cases where no convenient formulae could be obtained, the notation $= 0$ denotes reducibility, $=$ denotes equivalence and $\longrightarrow$ denotes one way identity.

* Das Gupta and Turnbull, "On the complete system of Linear Complexes," Proc. Edin. Math. Soc., 1929, *61-70*.

Application of the results in Table C

14 From the results in **Table C**, it is possible to get in the place of two factors two other factors one of which may associate with a third factor to give either a complete concomitant or in combination yield further equivalent forms

Illustrations

(i) $a_x(xKBLx)(aLb)(xALb)(ABx)$

$= a_x(bKLx)(aLBx)(xALb)(ABx)$, since $2'(11')=1'(12')$

$= a_x(bKLx)(aLb)(xALBx)(AB)$, since $(12')(1'2)=(11')(22')$

$= 0$

(ii) $(aKBa)(aKLx)(aKBx)a_\beta(aK\beta)$

$= (aKx)(aKBLa)(aKBx)a_\beta(aK\beta)$,

since $1(12'3)=(13)(12')+3(12'1)$

$= (aKx)(aKL\beta)(aKBx)a_\beta(aKBa)$,

since $(33')(12'1)=(13')(12'3)$

$= 0$, since $(12')(12'1)=0$.

(iii) $(aKx)(\beta Kx)(aKLa)a_\beta$

$= (aK/x)(\beta Kx)(aKLa)(a/\beta)$,

$= (aK\beta)(\beta Kx)(aKLa)(a_x)$, by §12

$= (aK\beta)(aKx)(aKL\beta)a_x$, since $3'(13)=1(33')+3(13')$.

15 In the following irreducible list given in Table D, only typical representative forms have been listed. Thus for instance where one complex K has found place, we understand that there is another similar form with complex L. Or again wherever forms are symmetrical with regard to either quadric, only one of the two has been retained, e.g., of the two forms

$$a_x(bKL\alpha)(aKb) \quad \text{and} \quad (aKL\alpha)b_x(aKb)$$

the former alone has been retained.

Another point to be noted is that this list excludes the invariants and covariants already given by Das Gupta in "The Simultaneous system of a quadric surface and two Linear Complexes" (Proc Lond Math Soc, Ser. 2, Vol 31, Part 7 and by Turnbull, in Proc Lond Math Soc, 2, **18** (1919), *69-94*.

TABLE D.

The irreducible invariants and covariants of two quaternary quadrics associated with two linear complexes.

14 Invariants —

$(AK)(BK)(AB)$,	$(aKb)a_\beta(aK\beta)b_\alpha$,
$(aKb)a_\beta(aK\beta)(bKL\alpha)$,	$(aKb)a_\beta(bKL\beta)$,
$(aKb)(aKL\beta)(bKL\beta)$;	$a_\beta(aKL\alpha)(aK\beta)$,
$(aKL\beta)(aKL\alpha)(aK\beta)$,	$(aKBL\alpha)(BK)$,
$(aKBL\alpha)(AB)(AK)$,	$(aKB\alpha)(BK)(aLb)b_\alpha$,
$(aKB\alpha)(BK)a_\beta(aK\beta)$,	$(aKB\alpha)(AB)(AK)(aKb)b_\alpha$;
$(aKBL\alpha)(BK)$,	$(aKBL\alpha)(AB)(AK)$

TABLE D—(*continued*).

121 Covariants :—

$a_x b_x (aKb)$; $\quad$ $a_x (bKLx)(aKb)$,

$(aKLx)(bKLx)(aKb)$; $\quad$ $a_x b_x (aBKx)(xB\alpha) b_\alpha$,

$(aKLx) b_x (aBKx)(xKBL\alpha) b_\alpha$, $\quad$ $a_x b_x (aBKx)(xKBL\alpha) b_\alpha$,

$(aKLx) b_x (aBKx)(aB\alpha) b_\alpha$, $\quad$ $a_x b_x a_\beta (bKL\beta)$,

$a_x b_x (aKL\beta)(bKL\beta)$, $\quad$ $a_x (bKLx) a_\beta (bKL\beta)$,

$a_x b_x a_\beta (\beta Ax)(bKAx)$, $\quad$ $a_x (bKLx) a_\beta (\beta Ax)(bKAx)$,

$(aKLx) b_x a_\beta (\beta Ax)(bKAx)$; $\quad$ $(aKLx)(bKLx) a_\beta (\beta Ax)(bKAx)$,

$a_x b_x a_\beta (xA\beta)(AB)(xB\alpha) b_\alpha$; $\quad$ $a_x (bKLx) a_\beta (xA\beta)(AB)(xB\alpha) b_\alpha$;

$a_x b_x a_\beta (\beta Ax)(AK)(BK)(xB\alpha) b_\alpha$;

$a_x (bKLx) a_\beta (\beta Ax)(AK)(BK)(xB\alpha) b_\alpha$;

$a_x b_x a_\beta (\alpha K\beta) b_\alpha$; $\quad$ $a_x b_x a_\beta (\alpha K\beta)(bKL\alpha)$,

$a_x b_x (aKL\beta)(\alpha K\beta)(bKL\alpha)$;

$a_x (bKLx) a_\beta (\alpha K\beta) b_\alpha$; $\quad$ $a_x (bKLx) a_\beta (\alpha K\beta)(bKL\alpha)$,

$a_x b_x (aKL\alpha) b_\alpha$; $\quad$ $a_x b_x (aKL\alpha)(bKL\alpha)$,

$(aKLx) b_x (aKL\alpha) b_\alpha$; $\quad$ $a_x (BK)(aKb)(bLAx)(AB)$;

$a_x (BK)(aKb) b_\alpha (\alpha K\beta)(\beta Ax)(AB)$,

$a_x (BK)(aKb)(bKL\alpha)(\alpha K\beta)(\beta Ax)(AB)$,

$a_x (BK)(aKb) b_\alpha (\alpha Kx)(\beta Kx)(\beta Ax)(AB)$,

$(aKLx)(BK)(aKb) b_\alpha (\alpha Kx)(\beta Kx)(\beta Ax)(AB)$;

$a_x (BK)(aBKx)$; $\quad$ $a_x (xKBLx)(aBKx)$,

TABLE D—(*continued*).

$(aKLx)(BK)(aBKx)$, $(aKLx)(xKBLx)(aBKx)$;

$a_x(BK)a_\beta(\beta Ax)(AB)$, $a_x(BK)a_\beta(xKAL\beta)(AB)$,

$(aKLx)(BK)a_\beta(\beta Ax)(AB)$; $a_x(BK)(aKL\beta)(\beta Ax)(AB)$,

$a_x(BK)a_\beta(\beta Ax)(AK)b_x b_a(aBx)$,

$(aKLx)(BK)a_\beta(\beta Ax)(AK)b_x b_a(aBx)$,

$(aKLx)(BK)a_\beta(\beta Ax)(AK)(bKLx)b_a(aBx)$,

$a_x(BK)a_\beta(aK\beta)(xBa)$, $a_x(BK)(aKL\beta)(aK\beta)(xBa)$,

$(aKLx)(BK)a_\beta(aK\beta)(xBa)$, $(aKLx)(BK)(aKL\beta)(aK\beta)(xBa)$;

$a_x(BK)a_\beta(bKL\beta)(bKAx)(AB)$; $a_x(BK)(aKLa)(aBx)$,

$(aKLx)(BK)(aKLa)(aBx)$, $a_x(\beta Kx)(aKb)b_a(aK\beta)$;

$a_x(\beta Kx)(aKb)(bKLa)(aK\beta)$; $(aKLx)(\beta Kx)(aKb)b_a(aK\beta)$,

$a_x(\beta Kx)a_\beta$, $a_x(\beta Kx)(aKL\beta)$, $(aKLx)(\beta Kx)a_\beta$,

$(aKLx)(\beta Kx)(aKL\beta)$; $a_x(\beta Kx)(aKLa)(aK\beta)$,

$a_x(aKx)(aKb)b_a$, $a_x(aKx)(aKb)(bKLa)$,

$(aKLx)(aKx)(aKb)b_a$, $a_x(aKx)(aKBx)(xBa)$;

$(aKLx)(aKx)(aKBx)(xBa)$, $a_x(aKx)a_\beta(\beta Ax)(bKAx)b_a$;

$a_x(aKx)a_\beta(\beta Ax)(AB)(xBa)$, $(aKLx)(aKx)a_\beta(\beta Ax)(AB)(xBa)$;

$a_x(aKx)a_\beta(aK\beta)$, $a_x(aKx)(aKLa)$;

$(aKLx)(aKx)(aKLa)$, $(AK)(BK)(xALBx)$;

$(xKALx)(BK)(AB)$,

$(AK)(BK)(\beta Ax)a_\beta a_x b_x b_a(xBa)$;

TABLE D—(*continued*)

$(AK)(BK)(\beta Ax)a_\beta(aKLx)b_x b_\alpha(xB\alpha)$,

$(AK)(BK)(\beta Ax)a_\beta(aKLx)(bKLx)b_\alpha(xB\alpha)$,

$(AK)(BK)(\beta Ax)(\alpha K\beta)(xB\alpha)$,

$(AK)(\beta Kx)(xKAb)(aKb)a_\beta$,

$(AK)(\beta Kx)(AB)(aKBx)(aKb)b_\alpha(\alpha K\beta)$,

$(AK)(\beta Kx)(AB)(aKBx)a_\beta$; $(AK)(\beta Kx)(AB)(B\alpha x)(\alpha K\beta)$,

$(AK)(\beta Kx)(\beta Ax)$, $(xKALx)(\beta Kx)(\beta Ax)$,

$(aKx)(\beta Kx)b_\alpha(aKb)a_\beta$, $(\alpha Kx)(\beta Kx)(xB\alpha)(AB)(\beta Ax)$,

$(\alpha Kx)(\beta Kx)(xB\alpha)(xALBx)(\beta Ax)$, $(\alpha Kx)(\beta Kx)(\alpha K\beta)$,

$(aKb)a_x(bKAx)(AK)$, $(aKb)(aKBx)(xB\alpha)b_\alpha$,

$(aKb)(aKBx)(AB)(AK)(\alpha Kx)b_\alpha$;

$(aKb)b_\alpha(xB\alpha)(AB)(AK)a_x$;

$(aKb)a_\beta b_\alpha(\alpha K\beta)(xA\beta)(AK)$;

$(aKb)(aKL\beta)b_\alpha(\alpha K\beta)(xA\beta)(AK)$;

$(aKb)a_\beta(bKL\alpha)(\alpha K\beta)(xA\beta)(AK)$,

$(aKBx)a_x(AB)(AK)$; $(aKBx)a_x(xALBx)(AK)$;

$(aKBx)(aKLx)(AB)(AK)$, $(aKBx)a_\beta(\alpha K\beta)(xB\alpha)$,

$(aKBx)(aKL\beta)(\alpha K\beta)(xB\alpha)$,

$(aKBx)a_\beta(\alpha K\beta)(\alpha Kx)(AB)(AK)$;

$a_\beta a_x(\beta Ax)(AK)$; $a_\beta(aKLx)(\beta Ax)(AK)$,

$(aKL\beta)(aKLx)(\beta Ax)(AK)$, $(aKL\beta)a_x(\beta Ax)(AK)$,

TABLE D—(*continued*).

$a_\beta a_x (xKAL\beta)(AK)$, $(aKL\alpha)(aKBx)(AB)(AK)(xK\alpha)$,

$(xB\alpha)a_\beta(AB)(AK)(aK\beta)a_x$,

$(aB\alpha)(aKL\beta)(AB)(AK)(aK\beta)a_x$;

$(AB)(AK)(xB\alpha)(aKx)$, $(xALBx)(AK)(xB\alpha)(aKx)$,

$(AB)(\beta Ax)(aK\beta)(xB\alpha)$, $(\beta Ax)(AK)(aK\beta)(aKx)$,

$(\beta Ax)(AK)(bKL\beta)b_x$,

$(aKBL\alpha)(aKBx)a_x$, $(aKBL\alpha)(aKBx)(aKLx)$,

$(aKBL\alpha)(xB\alpha)(aKx)$, $(aKBL\alpha)(xB\alpha)b_\alpha(aKb)a_x$,

$(aKB\alpha)a_x(BK)(aLx)$; $(aKB\alpha)a_x(aKBx)a_\beta(aK\beta)$;

$(aKB\alpha)(AB)(AK)a_x(aKx)$, $(aKB\alpha)(AB)(AK)(aKb)b_\alpha$,

$(aKBL\alpha)(aBx)(aKx)$;

$(aKBL\alpha)(xKBL\alpha)(aKx)$,

$(aKBL\alpha)(aBx)(aKL\alpha)a_x$.

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Some Hyperspace Harmonic Analysis Problems introducing Extensions of Mathieu's Equations

BY

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It is well known that two problems of harmonic analysis in ordinary three-dimensional space can be solved by Mathieu's functions, namely, (*a*) harmonic analysis for an orthogonal system of elliptic (or hyperbolic) cylinders,

$$x = \cos\xi \cosh\eta, \quad y = \sin\xi \sinh\eta, \quad z = z \qquad \ldots \quad (1)$$

(*b*) harmonic analysis for a system of confocal paraboloids,

$$\frac{x^2}{\lambda-1} + \frac{y^2}{\lambda} - 2z - \lambda = 0 \qquad \ldots \quad (2)$$

I have shown* that a similar problem in four-dimensional space, leads to the equation of *associated Mathieu's functions*,

$$\frac{d^2y}{dx^2} + 2n \cos x \frac{dy}{dx} + (a + k^2 \cos^2 x)y = 0,$$

when the change of variables, analogous to (1), is

$$x = \cos\xi \cosh\eta \cos\phi,\ y = \cos\xi \cosh\eta \sin\phi,\ z = \sin\xi \sinh\eta,\ t = t$$

introducing hypercylinders parallel to the t-axis, their bases in the xyz space being ellipsoids (or hyperboloids) of revolution. I now propose to show that some other hyperspatial change of variables, analogous to (2), leads to the same equation and also to another extension of Mathieu's equation.

* *Revue Scientifique* de l'Institut Mittag-Leffler, **X** (1932), *91*.

I.

Let us consider the change of variables

$$\frac{x^2}{\lambda-1}+\frac{y^2+z^2}{\lambda}-2t-\lambda = 0, \qquad (3)$$

$$y = z\cot\phi, \qquad (4)$$

which can be written,

$$x = \sqrt{(\rho-1)(\mu-1)(\nu-1)},$$

$$y = i\sqrt{\rho\mu\nu}\cos\phi,$$

$$z = i\sqrt{\rho\mu\nu}\sin\phi,$$

$$t = -\frac{\rho+\mu+\nu-1}{2},$$

denoting by ρ, μ, ν the three roots of (3), considered as an equation in λ. The hypersurfaces $\phi=$ constant are hyperplanes, and the three hypersurfaces ρ, μ, $\nu=$ constant are obtained by the revolution, out of their space of the confocal paraboloids (2).

The system being orthogonal, Laplace's equation is found, by the usual method, to be

$$\Delta U = \frac{1}{4}\frac{(\rho-\mu)(\mu-\nu)(\nu-\rho)}{\rho\mu\nu\sqrt{(\rho-1)(\mu-1)(\nu-1)}}\frac{\partial^2 U}{\partial\phi^2}$$

$$+\Sigma\frac{\mu-\nu}{\sqrt{(\mu-1)(\nu-1)}}\frac{\partial}{\partial\rho}\left[\rho\sqrt{\rho-1}\,\frac{\partial U}{\partial\rho}\right] = 0,$$

the summation symbol meaning that the unwritten terms are to be deduced from the written one by circular permutation of the letters ρ, μ, ν.

Let us try to solve this equation by assuming

$$U(\rho, \mu, \nu, \phi) = R(\rho)M(\mu)N(\nu)\cos m\phi$$

We then have

$$-\frac{m^2}{4}(\rho-\mu)(\mu-\nu)(\nu-\rho)\mathrm{RMN}$$

$$+\Sigma\mu\nu(\mu-\nu)\mathrm{MN}\rho\sqrt{\rho-1}\frac{d}{d\rho}\left[\rho\sqrt{\rho-1}\frac{d\mathrm{R}}{d\rho}\right] = 0$$

or

$$-\frac{m^2}{4}(\rho-\mu)(\mu-\nu)(\nu-\rho)$$

$$+\Sigma\mu\nu(\mu-\nu)\frac{\rho\sqrt{\rho-1}}{\mathrm{R}}\frac{d}{d\rho}\left[\rho\sqrt{\rho-1}\frac{d\mathrm{R}}{d\rho}\right] = 0.$$

But

$$-(\rho-\mu)(\mu-\nu)(\nu-\rho) = \mu\nu(\mu-\nu)+\nu\rho(\nu-\rho)+\rho\mu(\rho-\mu) = \Sigma\mu\nu(\mu-\nu)$$

Hence

$$\Sigma\mu\nu(\mu-\nu)\left[\frac{\rho\sqrt{\rho-1}}{\mathrm{R}}\frac{d}{d\rho}\left(\rho\sqrt{\rho-1}\frac{d\mathrm{R}}{d\rho}\right)+\frac{m^2}{4}\right] = 0.$$

Now h and k being arbitrary constants, we have

$$\Sigma h\rho\mu\nu(\mu-\nu) = \Sigma k\rho^2\mu\nu(\mu-\nu) = 0,$$

so that the equation can be written as

$$\Sigma\mu\nu(\mu-\nu)\left[\frac{\rho\sqrt{\rho-1}}{\mathrm{R}}\frac{d}{d\rho}\left(\rho\sqrt{\rho-1}\frac{d\mathrm{R}}{d\rho}\right)+\frac{m^2}{4}+h\rho+k\rho^2\right] = 0,$$

and the function $\mathrm{R}(\rho)$ must then satisfy the equation

$$\rho\sqrt{\rho-1}\frac{d}{d\rho}\left(\rho\sqrt{\rho-1}\frac{d\mathrm{R}}{d\rho}\right)+\left(\frac{m^2}{4}+h\rho+k\rho^2\right)\mathrm{R} = 0,$$

or

$$\rho^2(\rho-1)\frac{d^2\mathrm{R}}{d\rho^2}+(\tfrac{3}{2}\rho^2-\rho)\frac{d\mathrm{R}}{d\rho}+\left(\frac{m^2}{4}+h\rho+k\rho^2\right)\mathrm{R} = 0, \quad \ldots \quad (5)$$

the equations for M and N being exactly similar.

To reduce (5) to a known type, let us put

$$R(\rho) = \rho^{\frac{m}{2}} S(\rho).$$

We obtain

$$\rho(\rho-1)\frac{d^2S}{d\rho^2} + [(\tfrac{3}{2}+m)\rho-(m+1)]\frac{dS}{d\rho} + [h+\tfrac{1}{4}m(m+1)+k\rho]S = 0\,,$$

and if we make the change of variable, $\rho = \sin^2\theta$, we find

$$\frac{d^2S}{d\theta^2} + (2m+1)\cot\theta\,\frac{dS}{d\theta} - 4[h+k+\tfrac{1}{4}m(m+1)-k\cos^2\theta]S = 0,$$

the equation of Mathieu's associated functions, a solution of which can

be expressed in terms of the function $c\,e_n^{m+1/2}(\theta)$.

II.

Consider now the change of variables

$$x = \sqrt{(\rho-1)(\mu-1)(\nu-1)},$$

$$y = i\sqrt{\rho\mu\nu},$$

$$z = -\frac{\rho+\mu+\nu-1}{2},$$

$$t = t,$$

introducing hypercylinders having the confocal paraboloids (2) as bases in the xyz-space

Laplace's equation is

$$\frac{1}{4}\,\frac{\partial^2 U}{\partial t^2}\cdot\frac{(\rho-\mu)(\mu-\nu)(\nu-\rho)}{\sqrt{\rho\mu\nu(\rho-1)(\mu-1)(\nu-1)}}$$

$$+\sum\frac{\mu-\nu}{\sqrt{\mu\nu(\mu-1)(\nu-1)}}\,\frac{\partial}{\partial\rho}\left[\rho\sqrt{\rho-1}\,\frac{\partial U}{\partial\rho}\right] = 0,$$

or by taking

$$U = e^{\lambda t}R(\rho)M(\mu)N(\nu),$$

$$\frac{\lambda^2}{4}(\rho-\mu)(\mu-\nu)(\nu-\rho)+\Sigma(\mu-\nu)\left[\frac{\sqrt{\rho(\rho-1)}}{R}\frac{d}{d\rho}\left(\rho\sqrt{\rho-1}\;\frac{dR}{d\rho}\right)\right] = 0$$

But

$$(\rho-\mu)(\mu-\nu)(\nu-\rho) = -\rho^2(\mu-\nu)-\mu^2(\nu-\rho)-\nu^2(\rho-\mu) = -\Sigma\rho^2(\mu-\nu),$$

and as

$$\Sigma h(\mu-\nu) = \Sigma\kappa\rho(\mu-\nu) = 0,$$

we can write

$$\Sigma(\mu-\nu)\left[\frac{\sqrt{\rho(\rho-1)}}{R}\frac{d}{d\rho}\left(\rho\sqrt{\rho-1}\;\frac{dR}{d\rho}\right)+h+k\rho-\frac{\lambda^2}{4}\rho^2\right]=0$$

The equation for R is therefore

$$\rho(\rho-1)\frac{d^2R}{d\rho^2}+(\rho-\tfrac{1}{2})\frac{dR}{d\rho}+\left(h+\kappa\rho-\frac{\lambda^2}{4}\rho^2\right)R = 0$$

If we take $\rho=\cos^2\theta$, we obtain

$$-\tfrac{1}{4}\frac{d^2R}{d\theta^2}+\left(h+k\cos^2\theta-\frac{\lambda^2}{4}\cos^4\theta\right)R = 0,$$

which, since

$$\cos^4\theta=a\cos 4\theta+b\cos 2\theta+c,$$

$$\cos^2\theta=a'\cos 2\theta+b',$$

is of the type

$$\frac{d^2R}{d\theta^2}+(\alpha+\beta\cos 2\theta+\gamma\cos 4\theta)R = 0,$$

an extension of Mathieu's equation, but a particular case of Hill's.* As far as I know, it is the first time that such an equation, which occurs in some astronomical and physical problems, is found in a

* Analytical properties of this equation have been given by E. L. Ince, *Proc. Lond. Math. Soc.*, **XXI I** (1923).

potential question The same equation would occur* as it will appear immediately, when investigating solutions of the wave equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial v^2}{\partial z^2}=\frac{1}{c^2}\frac{\partial v^2}{\partial t^2},$$

in three-dimensional space, in a region bounded by the confocal paraboloids a solution being

$$v = R(\rho)M(\mu)N(\nu)e^{i\lambda ct}$$

* This fact was pointed out to me by Mr E T Copson.

Bull Cal Math Soc, Vol XXVII, Nos. 3 & 4 (1935).

On Reducible Hyperelliptic Integrals

BY

A. C Choudhury

(*Calcutta*)

The problem of determining hyperelliptic integrals reducible to elliptic integrals by the change of variables $x=\dfrac{U(t)}{V(t)}$, U and V being polynomials of degree $\geqq 3$ was considered by Goursat and others.* In this paper, I have discussed a method of getting such integrals and using polynomials of higher degree, I have got new reducible hyperelliptic integrals I, II, III, IV which seem to be not realised before

1. Suppose $$y=\frac{\phi(x)}{\psi(x)},$$

where $$\phi(x) = a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots+a_n,$$

$$\psi(x) = b_0x^n+nb_1x^{n-1}+\frac{n(n-1)}{1.2}b_2x^{n-2}+\ldots+b_n.$$

The discriminant $\Delta(y)$ of $\phi(x)-y\psi(x)=0$ is a polynomial of degree $2(n-1)$. Suppose that the roots of $\Delta(y)=0$, are $y_1, y_2, \ldots, y_k (k=2n-2)$ and $\phi(x)-y\psi(x)$ has only double roots corresponding to each of these quantities Then the double roots $x_1, x_2, \ldots, x_k$ are the branch points when we consider x as a function of y. We can expand $\phi(x)-y_i\psi(x)$ near x_i and since $y_i=\dfrac{\phi'(x_i)}{\psi'(x_i)}$, where dashes denote the differential coeffecient with respect to x,

* Bull de la Soc Math de Fr, t XIII.

$\phi(x)-y_i\psi(x)$

$$=\frac{(x-x_i)^2}{\psi'(x_i)}\left\{\frac{\phi''(x_i)\psi'(x_i)-\psi''(x_i)\phi'(x_i)}{1\ 2}+\right.$$

$$\left.+\frac{(x-x_i)^{n-2}}{n!}\left(\phi^n(x_i)\psi'(x_i)-\psi^n(x_i)\phi'(x_i)\right)\right\}$$

$$=\frac{(x-x_i)^2}{\psi'(x_i)}\chi(x, x_i),\ \text{say},\ (i=1, 2,.. k)$$

Take the product of all these identities for $i=1, 2, \ldots k$. Now x_i are the roots of $\phi'(x)\psi(x)-\psi'(x)\phi(x)=0$, therefore the product $R(x)$ of $(ab)_{01}^{n-1}\chi(x, x_1)\chi(x, x_2)\ldots\chi(x, x_k)$ is the eliminant of $\chi(x, z)=0$ and

$$\phi'(z)\psi(z)-\psi'(z)\phi(z)=0,\ (ab)_0,\ \text{being}\ (a_0b_1-a_1b_0)$$

Similarly the product

$$R_1\ \text{of}\ (ab)_{01}^{n-1}\psi'(x_1)\psi'(x_2)\ldots\psi'(x_k)$$

can be found. Therefore the discriminant

$$\Delta(y)=A\frac{(\phi'\psi-\phi\psi')^2 R(x)}{(ab)_{01}^{n-1}R_1\psi^k(x)},$$

where A is the coefficient of highest power of y in $\Delta(y)$. Thus

$$\int\frac{dy}{\sqrt{\Delta(y)}}=\sqrt{\frac{R_1}{A}(ab)_{01}^{n-1}}\int\frac{dx}{\sqrt{R(x).\ \psi^{1-k}(x)}}$$

and
$$\int\frac{dy}{\sqrt{y^{1-k}\Delta(y)}}=\sqrt{\frac{R_1}{A}(ab)_{01}^{n-1}}\int\frac{dx}{\sqrt{R(x)\ \phi^{1-k}(x)}}.$$

$\Delta(y)$ can be expanded in powers of y in the form $\Delta\ y^k+k\Theta_1y^{k-1}+\ldots+\Delta'$, where Δ is the discriminant of $\psi(x)$ alone, Δ' is the discriminant of $\phi(x)$ and $\Theta_1, \Theta_2, \ldots, \Theta_{k-1}$ invariants of $\phi(x)$ and $\psi(x)$ intermediate between Δ and Δ'. If we make some of these invariants zero, the conditions may be expressed in terms of the roots. Choose $\phi(x)$ and $\psi(x)$ such that the coefficients $\Delta, \Theta_1\ldots\Delta'$ are zero except any four consecutive invariants. Then

$$\int\frac{dy}{\sqrt{\Delta(y)}}=\sqrt{\frac{R_1}{A_1}(ab)_{01}^{n-1}}\int\frac{dx(q_0x^{k-2}+q_1x^{k-3}+\ldots)}{\sqrt{R(x)}},$$

where $q_0 x^{k-1} + q_1 x^{k-2}$ .. is the polynomial whose roots are those x, which are determined by the vanishing of the above invariants Then again if we make all of them zero except three consecutive quantities, $\Delta(y)$ will be a cubic expression giving us two reducible hyperelliptic integrals

$$\int \frac{dy}{\sqrt{\Delta(y)}} = \sqrt{M_1} \int \frac{N(x)\ dx}{\sqrt{R(x)\psi(x)}},$$

and
$$\int \frac{dy}{\sqrt{y\,\Delta(y)}} = \sqrt{M_1} \int \frac{N(x)dx}{\sqrt{R(x)\phi(x)}}$$

where M_1 is what $\frac{R_1}{A_1}\ (ab)_{0\,1}^{n-1}$ becomes in this case and $N(x)$ is a polynomial

2 We shall now consider particular involutions and take in the first instance, the case of a cubic one

$$y = \frac{\phi(x)}{\psi(x)} = \frac{a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3}{b_0 x^3 + 3b_1 x^2 + 3b_2 x + b_3}$$

Here

$\phi(x) - y_i \psi(x)$

$$= \frac{(x - x_i)^2}{b_0 x_i^2 + 2b_1 x_i + b_2}\ [x\{2(ab)_{01} x_i + (ab)_{02}\}$$
$$+ (ab)_{01} x_i^2 + 2(ab)_{02} x_i + 3(ab)_{12}],$$

$$\Delta(y) = y^4 \Delta + 4y^3 \Theta + 6y^2 \Phi + 4y\Theta' + \Delta',$$

where Δ and Δ' are discriminants of $\phi(x)$ and $\psi(x)$ and Θ, Φ, Θ' are obtained from Δ and Δ' by operating with

$$a_0 \frac{\partial}{\partial b_0} + a_1 \frac{\partial}{\partial b_1} + a_2 \frac{\partial}{\partial b_2} + a_3 \frac{\partial}{\partial b_3} \text{ or } b_0 \frac{\partial}{\partial a_0} + b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2} + b_3 \frac{\partial}{\partial a_3}$$

$R(x)$ is obtained by eliminating y between

$$(ab)_{01} y^4 + 2(ab)_{02} y^3 + \{(ab)_{03} + 3(ab)_{12}\} y^2 + 2(ab)_{13} y + (ab)_{23} = 0$$

and
$$(ab)_{01} y^2 + \{2(ab)_{02} + 2(ab)_{01} x\} y + \{(ab)_{02} x + 3(ab)_{12}\} = 0$$

and R_1 by eliminating y between

$$b_0y^2+2b_1y+b_2 = 0$$

$$(ab)_{01}y^4+2(ab)_{04}y^3+\{(ab)_{03}+3(ab)_{12}\}y^2+2(ab)_{13}y+(ab)_{23} = 0$$

When $\Delta=0$, $\psi(x)$ has a double root which may be taken to be either infinity or zero These are the cases as given by Goursat

2 Next take

$$y = \frac{\phi(x)}{\psi(x)} = \frac{a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4}{b_0x^4+4b_1x^3+6b_2x^2+4b_3x+b_4} .$$

In this case

$$\Delta(y) = \Delta y^6+6\Theta_1y^5+15\Theta_2y^4+20\Theta_3y^3+15\Theta_4y+\Delta',$$

where Δ is the discriminant of $\psi(x) = I^3-27J^2$,

$$I = b_0b_4-4b_1b_3+3b_2^2, \; J = b_0b_2b_4+2b_1b_2b_3-b_1b_3^2-b_4^2b_1-b_2^3,$$

and $\Delta' = I'^3-27J'^2$, I', J' being similar quantities for $\phi(x)$ If

$$\partial = a_0\frac{\partial}{\partial b_0}+a_1\frac{\partial}{\partial b_1}+a_2\frac{\partial}{\partial b_2}+a_3\frac{\partial}{\partial b_3}+a_4\frac{\partial}{\partial b_4},$$

$$\partial' = b_0\frac{\partial}{\partial a_0}+b_1\frac{\partial}{\partial a_1}+b_2\frac{\partial}{\partial a_2}+b_3\frac{\partial}{\partial a_3}+b_4\frac{\partial}{\partial a_4},$$

$$\Theta_1 = \partial\Delta = 3(I^2\partial I-18J\partial J),$$

$$\Theta_5 = \partial'\Delta' = 3(I'^2\partial' I'-18J'\partial' J').$$

There are three different ways of putting two conditions on $\phi(x)$ and $\psi(x)$

(a) $\Delta = 0, \quad \Theta_1 = 0,$

(b) $\Delta' = 0, \quad \Theta_5 = 0,$

(c) $\Delta' = 0, \quad \Delta = 0$

It is evident from the above value of Δ, Θ_1, that one simple way of making them zero is to put $I=0$, $J=0$ In this case the quartic $\psi(\;)$

has got a triple root which may be conveniently taken as zero or infinity Thus we get two substitutions,

$$y = \frac{a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4}{b_0 x^4 + 6b_1 x^3}$$

and
$$y = \frac{a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4}{4b_3 x + b_4}$$

One of these can be obtained from the other by changing x into $\frac{1}{x}$ and interchanging the coefficients

3 We shall construct two hyperelliptic integrals in these cases

Take
$$y = \frac{x^4+pq}{x^4+px^3}$$

$R(x)$, as obtained by eliminating z between

$$z^2+2x+3x^4 = 0,$$

and
$$z^4-4pz-3pq = 0,$$

is found to be

$$R(x) = 3q^2(27x^8-40px^5+14pqx^4+16p^2x^2-8p^2qx+3pq^2)$$

$$= 3q^2 R_1(x)$$

(I) Thus
$$\int \frac{dy}{\sqrt{27q^3y^4-16^2p(1-y)^3}} = \frac{1}{q}\int \frac{x^2dx}{\sqrt{R_1(x)}}$$

In the second case if we take $y=\frac{x^4+p}{x+q}$,

(II)
$$\int \frac{dy}{\sqrt{3^3y^4-4^4(p-qy)^3}} = \int \frac{dx}{\sqrt{3x^8-8qx^7+16q^2x^6+14px^4-40px^3+27p^2}}$$

Similar results are obtained when we make $\Delta'=0$, $\Theta_s=0$ by taking $I'=0$, $J'=0$ The hyperelliptic integrals in these cases are more readily obtained by changing y of the elliptic integrals in the preceding cases into $\frac{1}{y}$.

In the case when $\Delta=0$, $\Delta'=0$, both the quartics $\phi(x)$ and $\psi(x)$ have the double roots which must be different from each other. We shall find it convenient to take them zero and infinity. The substitution in these cases are

$$y = \frac{a_0 x^4 + 4a_1 x^3 + 6a_2 x^2}{6b_2 x^2 + 4b_3 x + b_4}$$

and
$$y = \frac{6a_2 x^2 + 4a_3 x + a_4}{b_0 x^4 + 4b_1 x^3 + 6b_2 x^2}$$

3. Next we shall consider the cases when $\Delta(y)$ can be reduced to a cubic expression. This can be obtained by putting three conditions on $\phi(x)$ and $\psi(x)$. There are four possibilities

$$(i)\quad \Delta=0,\quad \Theta_1=0,\quad \Theta_2=0,$$

$$(ii)\quad \Delta=0,\quad \Theta_1=0,\quad \Delta'=0,$$

$$(iii)\quad \Delta'=0,\quad \Theta_5=0,\quad \Theta_4=0,$$

$$(iv)\quad \Delta=0,\quad \Theta_3=0,\quad \Delta'=0,$$

From the expressions of Δ, Θ_1, Θ_2 in terms of I, J, ∂ I, ∂ J, we can see that one way of making $\Delta=0$, $\Theta_1=0$, $\Theta_2=0$ is to put $I=0$, $J=0$, $\partial J=0$. Making $I=0$, $J=0$, we get, when the triple root is taken to be zero,

$$y = \frac{a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4}{b_0 x^4 + 4b_1 x^3}$$

In this case $\partial J = -a_4 b_1^2$. We cannot take $a_4=0$, therefore $b_1=0$.

Hence $$y = \frac{a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4}{b_0 x^4}.$$

Similarly when the triple root is infinity,

$$y = \frac{a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4}{b_4},$$

which really amounts to $y = ax^4 + 4bx^3 + 6cx^2 + 4dx + e$, and the first one can be had by changing x into $\frac{1}{x}$ in this. This substitution can easily be reduced to

$$y = x^4 + 2ax^2 + 4bx.$$

Dolbnia * has given the hyperelliptic integral reducible by this transformation. Similarly, if we apply the conditions $\Delta'=0$, $\Theta_5=0$, $\Theta_4=0$, we have $\phi(x)$ reduced either to ax^4 or to a_4.

In case $\Delta'=0$, $\Delta=0$, we can take

$$y = \frac{a_0x^4+4a_1x^3+6a_2x^2}{6b_2x^2+4b_3x+b_4}$$

Hence $I'=3a_2^2$, $J'=-a_2^3$, $\partial I' = a_0b_4-4a_1b_3+6a_2b_2$,

$$\partial' J' = a_0a_2b_4+2a_1a_2b_3-b_4a_1^2-3a_2^2b_2$$

Therefore $\Theta_6=0$ gives

$$I'^2\,\partial I'-18J'\,\partial J' = 0 \text{ i.e. } b_4(3a_0a_2-2a_1^2) = 0$$

Now we can not put $b_4=0$, hence $3a_0a_2-2a_1^2 = 0$.

Thus $a_0x^2+4a_1x+6a_2$ should be a perfect square

$$y = \frac{x^2(x+a)^2}{6b_2x^2+4b_3x+b_4}$$

Similarly if we proceed with

$$y = \frac{6a_2x^2+4a_3x+a_4}{b_0x^4+4b_1x^3+6b_2x^2}$$

the condition $\Theta_6=0$ gives $3a_2a_4-2a_3^2=0$, that is, $6a_2x^2+4a_3x+a_4$ should be a perfect square.

We shall remain content by constructing the hyperelliptic integral in one of these cases. Take

$$y = \frac{(x+a)^2}{x^4+px^2}.$$

Eliminating y between

$$y^3+2ay^2+ap = 0$$

and

$$y^3+(3a+2x)y^2+y(2ax+x^2)+(ap+ax^2) = 0,$$

* *Bull. des Sciences Math.* 2^e *série*, t. XXV, 1901, pp. 114-116.

we get

$$R(x) = \begin{vmatrix} 2x+a & 2ax+x^2 & ax^2 \\ 2ax+x^2 & 3ax^2+4a^2x & a(2ax^2-2px-ap) \\ ax^2 & a(2ax^2-2px-ap) & -ap(2ax+x^2) \end{vmatrix},$$

$$R_1 = ap\left(ap+\frac{ap^2}{2}-\frac{p^3}{8}\right)$$

$$\Delta(y) = 2y^2[6p^4(p-3a^2)y^3-3p^2\{3p^3-2p(1+15a^2)-24a^2(1-18a^2)\}y^2$$

$$+2(4p^4-14a^2p^3-103a^4p+432a^6)y-6(p^2-4a^2p-9a^4)]$$

$$= 2y^2\Delta_1(y)$$

where $\Delta_1(y)$ denotes the expression within the crotchets

Therefore

$$\text{(III)} \qquad \int\frac{dy}{\sqrt{\Delta_1(y)}} = \sqrt{ap\left(ap+\frac{ap^2}{2}-\frac{p^3}{8}\right)}\int\frac{2x(x+a)\,dx}{\sqrt{R(x)}}.$$

Similar results hold good for the case when $\Delta=0$, $\Theta_1=0$, $\Theta_2=0$ and we shall not go into the details of this case

We shall conclude this paper by considering only one particular case of the involution of fifth order. Take

$$y = x^5-5ax$$

The discriminant in this case is $y^4-4^4a^5$, and

$$R(x) = x^{12}-18ax^8+113a^2x^4-256a^3$$

Thus we get

$$\text{(IV)} \qquad \int\frac{dy}{\sqrt{y^4-4^4a^5}} = 5\int\frac{dx}{\sqrt{R(x)}}.$$

25

On Appell's Function $P(\theta, \phi)$

BY

Maurice de Duffahel

(*Stamboul*)

1. Appell's functions, $P(\theta, \phi)$, $Q(\theta, \phi)$ and $R(\theta, \phi)$ are defined by the expansion*

$$e^{j\theta+j^2\phi}=P(\theta, \phi)+jQ(\theta, \phi)+j^2R(\theta, \phi),$$

where $j^3=1$, affording, both for the third order and the field of two variables, a very direct generalisation of the circular functions, as

$$e^{i\theta} = \cos\theta+i\sin\theta$$

They can be written as follows.

$$P(\theta, \phi) = \frac{1}{3}\left(e^{\theta+\phi} + e^{j\theta+j^2\phi} + e^{j^2\theta+j\phi} \right),$$

$$Q(\theta, \phi) = \frac{1}{3}\left(e^{\theta+\phi} + j^2e^{j\theta+j^2\phi} + je^{j^2\theta+j\phi} \right),$$

$$R(\theta, \phi) = \frac{1}{3}\left(e^{\theta+\phi} + je^{j\theta+j^2\phi} + j^2e^{j^2\theta+j\phi} \right),$$

and they satisfy the fundamental relation,

$$P^3+Q^3+R^3-3PQR = 1$$

* Comptes Rendus de l'Acad des Sciences de Paris, **84** (1877), *540*.

I showed recently that they are of great help in solving numerous problems connected with the equation,

$$\Delta_3 v = \frac{\partial^3 v}{\partial x^3}+\frac{\partial^3 v}{\partial y^3}+\frac{\partial^3 v}{\partial z^3}-3\frac{\partial^3 v}{\partial x\partial y\partial z} = 0,$$

and allied equations.*

The object of this short note is to state some elementary remarks on the function $P(n\theta, n\phi)$ where n is an integer, and to make more conspicuous the analogy between it and $\cos n\theta$.

2. Let us consider the expression

$$E = \log\,(1-ae^{\theta+\phi})(1-ae^{j\theta+j^2\phi})(1-ae^{j^2\theta+j\phi}),$$

where a is an arbitrary constant and try to expand it in ascending powers of a. We have

$$E = \log\,(1-ae^{\theta+\phi})+\log\,(1-ae^{j\theta+j^2\phi})+\log\,(1-ae^{j^2\theta+j\phi})$$

$$= -\sum_n \frac{a^n e^{n(\theta+\phi)}}{n} - \sum_n a^n\frac{e^{n(j\theta+j^2\phi)}}{n} - \sum_n a^n\frac{e^{n(j^2\theta+j\phi)}}{n}$$

$$= -\sum 3a^n P(n\theta, n\phi)/n.$$

Now as $1+j+j^2=0$, we can write

$$E = \log\,[1-ae^{\theta+\phi}-ae^{j\theta+j^2\phi}-ae^{j^2\theta+j\phi}+a^2e^{-j^2\theta-j\phi}$$
$$+a^2e^{-\theta-\phi}-a^3]$$

$$= \log\,[1-3aP(\theta, \phi)+3a^2P(-\theta, -\phi)-a^3].$$

So when we obtain the function $P(n\theta, n\phi)$ through the generating function,

$$-\log\,[1-3aP(\theta, \phi)+3a^2P(-\theta, -\phi)-a^3],$$

the co-efficient of a^n being $3P(n\theta, n\phi)/n$.

* Bulletin de Math. Supér., année, **38** (1932-33), *128*; *Cf.* Y. Devisme, Comptes Rendus, **193** (1931), *981*.

The noteworthy analogy with the circular functions arises from the fact that the co-efficient of a^n in the expansion of

$$-\log\left[1-2a\cos\theta+a^2\right]$$

is

$$(2\cos n\theta)/n$$

3. The expansion just obtained,

$$-\log\left[1-3aP(\theta,\phi)+3a^2P(-\theta,-\phi)-a^3\right] = \sum_n 3a^n\frac{P(n\theta, n\phi)}{n},$$

shows that $P(n\theta, n\phi)$ can be expressed as a polynomial with respect to $P(\theta,\phi)$ and $P(-\theta,-\phi)$

We observe that

$$P(-\theta,-\phi) = P^2(\theta,\phi)-Q(\theta,\phi)R(\theta,\phi),$$

so that $P(n\theta, n\phi)$ is a polynomial with respect to P and QR. For instance,

$$P(2\theta, 2\phi) = P^2+2QR = 3P^2(\theta,\phi)-2P(-\theta,-\phi),$$

$$P(3\theta, 3\phi) = 1+9PQR = 9P^3(\theta,\phi)-9P(\theta,\phi)P(-\theta,-\phi)+1.$$

Our expression leads readily to the following general result:

$$\frac{P(n\theta, n\phi)}{n} = \sum_p\sum_q \frac{(-1)^q 3^{p+q}}{n+2p+q}\,{}_pC_{(n+2p+q)/3}\;{}_qC_{(n+q-p)/3}\,P^p(\theta,\phi)P^q(-\theta,-\phi),$$

with $p \leq n$, $q \leq \frac{1}{2}(n-p)$. The symbol ${}_sC_r$ stands for the number of combinations of r objects s at a time. Of course, $\frac{1}{3}(n+2p+q)$ and $\frac{1}{3}(n+q-p)$ must be positive integers.

4. Similar formulæ can, of course, be written for $Q(n\theta, n\phi)$ and $R(n\theta, n\phi)$

We may use the relations,

$$Q(-\theta,-\phi) = Q^2-RP,$$

$$R(-\theta,-\phi) = R^2-PQ$$

If we take $\phi=0$, the function P reduces to one of the *sines of the third order*,

$$f_1(\theta) = \tfrac{1}{3}(e^{\theta}+e^{j\theta}+e^{j^2\theta}),$$

and we obtain the expansion,

$$-\log\,[1-3af_1(\theta)+3a^2f_1(-\theta)-a^3] = \sum_n 3a^n f_1(n\theta)/n,$$

showing that $f_1(n\theta)$ can be expressed as a polynomial with respect to $f_1(\theta)$ and $f_1(-\theta)$

We may, perhaps, suggest the following researches :

(*a*) To express $P(n\theta, n\phi)$ as a hypergeometric function of two variables and of the third order (one of the functions introduced by Y Kampe de Feriet)

(*b*) To extend the result to sines of the 4th, etc., order, with one or two variables

(*c*) To find a generating function for $P(h\theta, h\phi)$

Bull. Cal. Math. Soc., Vol. XXVII, Nos. 3 & 4 (1935).

REVIEW

Text-book of Spherical Trigonometry.—By P. N. Mitra, M.A., pp. xxii + 163 (1935) (Calcutta University Press).

This book, which is intended for the use of Post-Graduate students, covers a field which is well defined by tradition and has been thoroughly explored by many authors. Little that is novel is therefore expected.

The author begins with a historical introduction tracing the development of the subject from the time, when the study of scientific astronomy began. The author goes to prove that the subject was known to the Hindu Astronomers, long before this date, and the fundamental formulæ are of Indian origin. On the whole, the history is very instructive and interesting too.

The first two chapters of the book deal with the preliminary definitions and propositions and in Chapters III and IV, some of the fundamental propositions have been established. The treatment in these chapters are similar to those given by previous authors.

In Chapters V, VI, VII, some theorems concerning the properties of a spherical triangle have been established. In articles 5, 10 and the following, the author uses the term "sine of the triangle," but while giving the definition in article 3, he defines it as the "norm of the sides of the spherical triangle," only giving a reference in the foot-note. The author would do well if he would define $2n$ as the "sine of the triangle" and place it in the body of the book, instead of giving it in the foot note.

Any text-book on Trigonometry must consist of a large number of examples worked and unworked. The worked out examples in the book under review are illustrative. There is a good collection of examples after each chapter. But it is advisable that some elementary problems on Spherical Astronomy be introduced, in order to illustrate

the application of the subject, which, in the opinion of the Reviewer, will make the subject more intersting

The printing of the book specially on the last part is defective' with a few misprints. The appearance of some of the pages is spoiled by the use of broken types and also for want of symmetry in spacing

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